<u>Mesoscale Meteorology: Density Currents</u>

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Introduction

A *density current* is defined as the intrusion of a denser fluid beneath a lighter fluid. Consider the hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho g$$

Let us consider a density current of finite depth, such that isobaric surfaces are parallel to constant height surfaces above the density current. Evaluated between the ground ($z = 0$) and the top of the density current, the denser fluid will have a larger decrease in pressure over the depth of the density current than the lighter fluid. For fixed pressure at the top of the density current, this implies the presence of a *mesohigh* at the surface within the density fluid. (This is the same as a hypsometric argument, where a colder layer is associated with reduced thickness and thus higher pressure below the layer.) This establishes a horizontal pressure gradient force directed away from the mesohigh that is responsible for density current motion and displacing the lighter fluid above the denser fluid.

While much of the pressure differential across a density current results from the above hydrostatic principles, there are substantial non-hydrostatic contributions along the leading edge of the density current and within the density current beneath the strongest downdraft. Further, a *wake low*, which forms due to compressional warming associated with unsaturated descent atop the density current, may be found rearward of the mesohigh. In such a case, the horizontal pressure gradient force in the rear of a density current may be elevated over that along the leading edge.

Like sea breezes, which are a subclass of density current, density currents typically have a head on their leading edge, with depth that can be as much as twice as large as that behind its leading edge. It is along the head where the denser fluid displaces the less dense fluid upward. The depth of this lifting depends not just on the density difference between the two fluids but also on the ambient vertical wind shear and whether the ambient flow opposes or is along density current motion. The connection to the ambient vertical wind shear will be discussed in greater detail when we discuss mesoscale convective systems later this semester and briefly later in this lecture.

Whereas air parcels often flow *through* a frontal zone, even as they conserve (equivalent) potential temperature, air parcels generally flow *over* a density current. In this sense, density currents behave as *material surfaces*; i.e., density current air is to large extent separated from its surroundings. This is not to imply that, once generated, density current air remains unmodified in perpetuity; diabatic processes are important contributors to density current formation and modification, and instability-driven mixing across the density current can weaken density currents while also contributing to a complex three-dimensional internal structure. Density currents are typically about 1 km deep, with some as shallow as a few hundred meters and some as deep as a few kilometers.

Representative meteorological examples of density currents include sea and land breezes, locally-strong cold fronts where the Coriolis force does not offset the density-difference-driven horizontal pressure gradient force (neglecting friction), and *outflow boundaries*. Outflow boundaries separate the ambient environment from air cooled primarily by evaporation of falling hydrometeors within subsaturated air. Where frozen hydrometeors such as snow, ice, graupel, and hail are present within a precipitating feature, cooling resulting from melting and sublimation can also act to cool the air. In general, precipitation-cooled air behind an outflow boundary is coldest when mid-tropospheric air is relatively cool and moist, to promote a large hydrometeor mass, while air within the planetary boundary layer is relatively dry, to promote efficient evaporation of falling hydrometeors.

Three-Dimensional Structure

Salient density current dynamics are two-dimensional – one horizontal and one vertical dimension. However, density currents have distinct three-dimensional structure in the form of lobes and clefts.

Consider a propagating density current. Though it is convenient to consider both denser and lighter fluids as having constant density, this is an oversimplification. Due to its inverse relationship with buoyancy, the denser fluid is most dense at ground level, with decreasing density at higher altitudes until the top of the density current. Thus, the horizontal pressure gradient force is largest at the surface (when neglecting friction) and decreases in magnitude upward from there. This implies more rapid propagation at the ground in the absence of friction.

When friction is considered, propagation at the ground is slowed, allowing for the density current just above the ground to propagate as or more rapidly than at the ground. This results in the development of a shallow vertical layer over which potential temperature decreases with height, a statically unstable situation and one that promotes turbulence – i.e., vertical mixing – along the density current's leading edge. This is known as *lobe and cleft instability* and results in the development of lobes and clefts along a density current's leading edge. Billows along the top of a density current can result from Kelvin-Helmholtz instability, also like sea breezes.

Mathematical Development

We now wish to consider density current dynamics in a simplified two-dimensional framework. To do so, we neglect friction and assume a flat surface, no ambient buoyancy, no ambient vertical wind shear, and (for simplicity) no ambient flow.

We start by deriving an expression for the surface pressure perturbation associated with a density current. Assuming a spherical Earth, neglecting the vertical component of the Coriolis force, and neglecting friction, the vertical momentum equation can be written as:

$$\rho \frac{dw}{dt} = \frac{\partial p}{\partial z} - \rho g$$

For pressure $p = \overline{p} + p'$ and density $\rho = \overline{\rho} + \rho'$, if we define a horizontally-homogeneous base state in hydrostatic balance, i.e.,

$$0 = -\frac{\partial \overline{p}}{\partial z} - \overline{\rho} g$$

and subtract it from the vertical momentum equation, we obtain:

$$\rho \frac{dw}{dt} = -\frac{\partial p'}{\partial z} - \rho' g$$

If we divide through by ρ, we obtain:

$$\frac{dw}{dt} = -\alpha \frac{\partial p'}{\partial z} - \frac{\rho'}{\rho} g$$

where $\alpha = \rho^{-1}$. Substituting with the definition of buoyancy,

$$B = -\frac{\rho'}{\rho} g$$

we obtain:

$$\frac{dw}{dt} = -\alpha \frac{\partial p'}{\partial z} + B$$

Letting $\alpha = \alpha_0$, a constant specific volume characterizing the mean conditions, and assuming no vertical parcel accelerations, we can write:

$$\alpha_0 \frac{\partial p'}{\partial z} = B$$

If we integrate this equation from $z = 0$ to $z = H$, where H is density current depth, at a location in the density current itself, we obtain:

$$\int_{p_0'}^{p_H'} \alpha_0 \partial p' = \int_0^H B \partial z$$

$$\alpha_0 p_H' - \alpha_0 p_0' = -gH \frac{\rho_2 - \rho_1}{\rho_1}$$

where we have assumed that the density in the ambient air is equal to ρ_1, the density in the density current is equal to ρ_2, and that both are constant with height (e.g., B in the density current is constant with height) to evaluate the right-hand side integral. If the pressure perturbation at density current top is equal to zero, which is admittedly a poor approximation, then we can write:

$$\alpha_0 p_0' = gH \frac{\rho_2 - \rho_1}{\rho_1}$$

From this, the surface pressure perturbation is larger for a deeper, denser (colder) density current.

We next derive an expression for the density current's forward motion. While the course textbook presents two methods for doing so, we focus only on the second of the two because it is both more general in that it does not assume constant density over the depth of the density current. Further, the second derivation uses a slightly simplified version of the same framework by which mesoscale convective system structure can be understood. Further, note that in contrast to sea breezes, where we derived an expression for the *change* in circulation (or, with an additional step, velocity) around the perimeter of the density-driven circulation, here we consider only the density current's motion.

Recall the definition of vorticity:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\mathbf{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k} = \xi + \eta + \zeta$$

The components of the vorticity vector define rotation in the *y-z*, *x-z*, and *x-y* planes, respectively. Consider a two-dimensional density current in the *x-z* plane, such that we are interested in η. A time-tendency equation for η can be obtained by taking $\mathbf{j} \cdot \nabla \times \left(\frac{\partial \mathbf{v}}{\partial t} \right)$; i.e., by taking the curl of the momentum equation and finding its component along the **j** unit vector.

This equation takes the form:

$$\frac{\partial \eta}{\partial t} = -\vec{\mathbf{v}} \cdot \nabla \eta + \vec{\omega} \cdot \nabla v + f \frac{\partial v}{\partial z} - \frac{\partial B}{\partial x} + \mathbf{j} \cdot \left(\nabla \times \vec{\mathbf{F}} \right)$$

The right-hand side terms indicate advection of η, stretching and tilting of η, tilting of f, baroclinic generation, and friction. Noting that we are in a two-dimensional (x,z) plane, $v = 0$ and thus terms involving v can be neglected. Further, if we neglect friction, we obtain:

$$\frac{\partial \eta}{\partial t} = -\vec{\mathbf{v}} \cdot \nabla \eta - \frac{\partial B}{\partial x}$$

If the advection term is encapsulated into the local derivative on the left-hand side, this equation states that changes in η following the motion result from horizontal buoyancy gradients, where:

$$B = -\frac{\rho'}{\rho} g \approx \frac{\theta_v'}{\overline{\theta_v}} g$$

where the approximation assumes no hydrometeors. If we make the Boussinesq approximation, then the continuity equation can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

If we multiply this equation by η, we obtain:

$$\eta \frac{\partial u}{\partial x} + \eta \frac{\partial w}{\partial z} = 0$$

Note that:

$$\frac{\partial}{\partial x}(u\eta)+\frac{\partial}{\partial z}(w\eta)=u\frac{\partial \eta}{\partial x}+\eta\frac{\partial u}{\partial x}+w\frac{\partial \eta}{\partial z}+\eta\frac{\partial w}{\partial z}$$

From the continuity equation, the second and fourth terms of the above are zero. The first and third terms are simply the expansion of $-\vec{\mathbf{v}}\cdot\nabla\eta$ in the two-dimensional plane. Thus, we can substitute to obtain:

$$\frac{\partial \eta}{\partial t}=-\frac{\partial}{\partial x}(u\eta)-\frac{\partial}{\partial z}(w\eta)-\frac{\partial B}{\partial x}$$

This is what is known as the flux form of the time tendency equation for η, as the advection terms are written in terms of flux terms. Here, the u considered is that relative to density current's motion.

Let us consider a two-dimensional control volume, with horizontal ends $x = L$ and $x = R$ that are placed well behind and well ahead of the density current's leading edge and vertical bounds $z = 0$ and $z = d$ that are placed at the ground and well above the density current's top. If we integrate this equation over the control volume, we obtain:

$$\frac{\partial}{\partial t}\left(\int_0^d\int_L^R \eta\,dxdz\right)=-\int_0^d\int_L^R \frac{\partial}{\partial x}(u\eta)dxdz-\int_L^R\int_0^d \frac{\partial}{\partial z}(w\eta)dzdx-\int_0^d\int_L^R \frac{\partial B}{\partial x}dxdz$$

If we take the innermost integral for each right-hand side term, we obtain:

$$\frac{\partial}{\partial t}\left(\int_0^d\int_L^R \eta\,dxdz\right)=-\int_0^d\left((u\eta)_R-(u\eta)_L\right)dz-\int_L^R\left((w\eta)_d-(w\eta)_0\right)dx-\int_0^d\left(B_R-B_L\right)dz$$

Since we assumed the ambient buoyancy to be equal to zero, then $B_R = 0$. For no vertical motion across the ground surface, $w_0 = 0$. This allows us to write:

$$\frac{\partial}{\partial t}\left(\int_0^d\int_L^R \eta\,dxdz\right)=-\int_0^d\left((u\eta)_R-(u\eta)_L\right)dz-\int_L^R\left((w\eta)_d\right)dx+\int_0^d\left(B_L\right)dz$$

If we assume purely horizontal flow at $z = d$, well above the top of the density current, then $w_d = 0$. Further, if we assume a steady-state solution (e.g., neglecting factors that could cause the density current to strengthen or weaken), then the time-tendency term is zero. Thus, we obtain:

$$0=-\int_0^d(u\eta)_R\,dz+\int_0^d(u\eta)_L\,dz+\int_0^d\left(B_L\right)dz$$

where we have separated the integral of the $u\eta$ terms into its components. At $x = L$ and $x = R$, the edges of our control volume, the flow is nearly horizontal at all altitudes. In other words, it is only along the density current's leading edge where w and its zonal variation are appreciably non-zero. Thus,

$$\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\approx\frac{\partial u}{\partial z}$$

This allows us to write:

$$0 = -\int_0^d \left(u\frac{\partial u}{\partial z} \right)_R dz + \int_0^d \left(u\frac{\partial u}{\partial z} \right)_L dz + \int_0^d (B_L) dz$$

Or, equivalently,

$$0 = -\int_0^d udu_R + \int_0^d udu_L + \int_0^H (B_L) dz$$

where because $B = 0$ in the ambient environment, such as above the top of the density current, the buoyancy term integral has equivalently been written over the range 0 → H, where H is the density current's depth.

If we complete the first two integrals, we obtain:

$$0 = -\frac{1}{2}\left(u_{d,R}^2 - u_{0,R}^2\right) + \frac{1}{2}\left(u_{d,L}^2 - u_{0,L}^2\right) + \int_0^H (B_L) dz$$

Recall that we defined u as being the velocity relative to the density current's motion. If we place $x = L$ at a point where it has no velocity relative to the density current's motion, then $u_{0,L} = 0$. For no ambient vertical wind shear, $u_{d,R} = u_{0,R}$. Together, these allow us to write:

$$u_{d,L}^2 = -2\int_0^H (B_L) dz$$

Since $u_{0,L} = 0$, and given the simplified definition of η, this equation gives a quantitative measure of the horizontal vorticity generated along the density current's leading edge due to buoyancy and, in particular, negative buoyancy and its vertical variation within the density current. The horizontal vorticity is a measure of the circulation along the density current's leading edge, with flow directed toward the ambient air at the ground, away from the ambient air at the top of the density current, and ascending along the density current's leading edge. If the air within the density current is more negatively buoyant (e.g., denser, generally implying colder) and/or over a deeper vertical extent, the baroclinically-generated horizontal vorticity is larger.

Recall that because there is no ambient vertical wind shear, $u_{d,R} = u_{0,R}$. If $z = d$ is sufficiently above the density current's top so as to be unaffected by the density current, and given no ambient flow, then we can assume that $u_{d,L} = u_{d,R}$. This allows us to write:

$$u_{d,L}^2 = u_{0,R}^2 = -2\int_0^H (B_L) dz$$

Or, taking the square root:

$$u_{0,R} = -\sqrt{-2\int_0^H (B_L)dz}$$

where we have taken the negative root assuming an eastward-propagating density current, such that the ambient flow relative to the density current is directed toward the west (i.e., zero ambient flow minus a positive current propagation speed is negative, indicating westward-directed current-relative wind). The above velocity is thus viewed as the *speed of the density current relative to the ambient flow*.

What about in the case of non-zero ambient flow at the ground? Let the ground-relative ambient flow have velocity $U_{0,R}$. Further, let the density current propagation velocity relative to the ground have velocity U_c. By definition, $u_{0,R} = U_{0,R} - U_c$, or the speed of the density current relative to the ambient flow is equal to the ground-relative ambient flow minus the ground-relative speed of the density current. This allows us to write:

$$U_C = U_{0,R} + \sqrt{-2\int_0^H (B_L)dz}$$

Thus, the ground-relative speed of the density current is equal to the ambient flow plus a buoyancy-related term. Eastward-propagation is faster for greater eastward ambient flow and a colder and/or deeper density current; the opposite is true for greater westward ambient flow and a warmer and/or shallower density current. In addition to unsheared ambient flow, ambient vertical wind shear can modulate the nature of lifting along the density current's head. The precise impact of vertical wind shear depends upon ambient shear strength, ambient shear depth relative to density current's depth, and density current strength. In general, lifting is deepest when the vertical wind shear has opposite direction to density current propagation, vertical wind shear depth is approximately 3/4 that of the density current, and when the ground-relative wind is equal to but in the opposite direction of the density current propagation speed. We will consider these concepts in more detail when discussing mesoscale convective systems later in the semester.

Returning to the expression for the speed of the density current relative to the ambient flow, if we assume that the minimum buoyancy B_{min} in the density current is found at the ground and decreases linearly to zero over the density current's depth, then integrated buoyancy over the density current is equal to $0.5B_{min}*H$. Thus, we obtain:

$$u_{0,R} = -\sqrt{-B_{\min}H}$$

Substituting for B_{min} with the definition of buoyancy, we obtain:

$$u_{0,R} = -\sqrt{-\frac{\theta'_{v0}gH}{\overline{\theta_v}}}$$

where θ'_{v0} is the virtual potential temperature perturbation at level of minimum buoyancy (i.e., the surface) and $\overline{\theta_v}$ is the mean virtual potential temperature at the surface within the ambient air. For a density current of 1 km depth, with θ'_{v0} = -10 K and $\overline{\theta_v}$ = 300 K, each representing common values of evaporatively-driven outflows or cold pools, $u_{0,R}$ = 18 m s^{-1}, which compares well with both observations (assuming no ambient flow) and numerical simulations of density currents.