Appendix A: Defining the Yanai Apparent Heat Source Q_1 and Apparent Moisture Sink Q_2

The spatial distribution of net heating across the tropics drives motion within the tropical latitudes. Before showing this using a simple dynamical model of the atmosphere, however, we first need to use basic principles of thermodynamics to introduce the concept of heat sources and heat sinks.

We first define dry static energy s and moist static energy h as follows:

(A1) $$s = c_p T + gz$$

(A2) $$h = c_P T + gz + L_v q$$

where c_P, g, and L_v are constants, T is temperature, z is height, and q is the mixing ratio of water vapor. The dry static energy is a measure of enthalpy plus potential energy and the moist static energy is a measure of dry static energy plus latent energy (e.g., energy released as water changes phases).

Dry static energy is approximately conserved following the motion for dry-adiabatic processes. Moist static energy is approximately conserved following the motion for dry- and moist-adiabatic processes. However, the presence of diabatic processes leads to neither dry nor moist static energy being conserved. In particular, the first law of thermodynamics states that changes in dry static energy *following the motion* are a function of the heating rate. In the troposphere, this heating is a function of the net radiation (e.g., incoming minus outgoing) and latent heating (particularly condensation and evaporation, as these are associated with much greater latent heats than phase changes between liquid and solid water).

In its simplest form, the *apparent heat source* Q_1 (positive for increased dry static energy) is defined by this heating rate, such that:

(A3) $$Q_1 = \frac{Ds}{Dt} = Q_R + L_v(c - e)$$

where Q_R is the heating rate due to radiation, c is the rate of condensation per unit mass of air, and e is the rate of evaporation of cloud droplets per unit mass of air.

In (A3), Q_1 is defined with respect to s, which is applicable independent of scale. However, we wish to instead derive an expression for $\overline{s}$, or the contribution to s from only large-scale processes (or those that we can readily measure, in contrast to the turbulent scales which we cannot readily measure). We can write any variable as the sum of a large-scale mean (overbar) and local perturbation (prime), such as:

$$s = \overline{s} + s'$$

Further, the total derivative can be written in flux form, where:

$$\frac{D(\quad)}{Dt} = \frac{\partial(\quad)}{\partial t} + \nabla \cdot \left(\boldsymbol{v}(\quad)\right)$$

The flux form of the total derivative is identical to the advection form of the total derivative so long as the continuity equation holds.

If we write s, c, and e in terms of their large-scale mean and local perturbation quantities, expand the total derivative, apply Reynolds' averaging to separate the large (resolvable) and small (unresolvable) scales of motion, and apply Reynolds' postulates to simplify the result, we obtain:

(A4a) $$Q_1 = \frac{\partial \bar{s}}{\partial t} + \nabla \cdot \left(\overline{\boldsymbol{v}'s'}\right) = \overline{Q_R} + L_v(\bar{c} - \bar{e})$$

The flux-form term in the middle of this equation has three components: two horizontal ($u's'$ and $v's'$) and one vertical ($\omega's'$). These represent the transport of perturbation dry static energy by turbulent eddies. If we assume that the horizontal components of this term can be neglected, we can rewrite (A4a) as:

(A4b) $$Q_1 = \overline{Q_R} + L_v(\bar{c} - \bar{e}) - \frac{\partial}{\partial p}(\overline{s'\omega'})$$

The third right-hand-side term of (A4b) represents the large-scale average of the turbulent *vertical* sensible heat transport on the small, or unresolvable, scales of motion. In the aggregate, (A4) shows that the apparent large-scale heating is a function of large-scale radiative heating, the large-scale release of latent heat due to net condensation, and the large-scale-averaged turbulent vertical sensible heat transport. Because the left-hand side of (A4b) is related to s, Q_1 can be related to changes in temperature or, through Poisson's law, potential temperature, following the motion.

Vertically integrating (A4b) between the surface (p_{sfc}) and tropopause (p_{trop}), we obtain:

(A5) $$< Q_1 >=< Q_R > +L_v P + S$$

where P is the precipitation rate (where the net condensation is assumed to fall out as precipitation), S is the vertical surface sensible heat flux, and brackets represent vertically integrated quantities. This equation enables us to readily demonstrate the impact of diabatic processes upon the vertically integrated horizontal heating distribution.

We can also define an equation of moisture continuity as follows (Yanai et al. 1973, their Eqn. 7):

(A6) $$\frac{Dq}{Dt} = e - c$$

where the rate of change of water vapor mixing ratio q following the motion is a function of net evaporation. If we multiply (A6) by L_v, re-arrange the right-hand side, use the flux form of the total derivative, and apply Reynolds' averaging, we obtain an expression for Q_2, the *apparent moisture sink*:

(A7) $$Q_2 = -L_v \frac{D\overline{q}}{Dt} = L_v(\overline{c} - \overline{e}) + L_v \frac{\partial}{\partial p}\left(\overline{q'\omega'}\right)$$

As the name moisture <u>sink</u> implies, Q_2 is positive for a <u>reduction</u> of q (or reduction in moist static energy) following the motion. The last term on the right-hand side of (A7) represents the large-scale average of the turbulent vertical latent heat transport on smaller scales.

Next, we vertically integrate (7) to obtain a relationship for $<Q_2>$ similar to that for $<Q_1>$ in (A6):

(A8) $$< Q_2 >= L_v(P - E)$$

where E is the surface evaporation rate per unit area, commonly referred known as the vertical surface latent heat flux.