Appendix B: Derivation of the Equatorial Wave Modes Using the Shallow-Water Equations

For the shallow-water system, there are three relevant equations. The first, the equation of motion, describes the two-dimensional *(x,y)* motion of the system. The second, the hydrostatic equation, describes the nature of vertical motions (here free of vertical parcel accelerations) within the system. Together, these equations describe the conservation of momentum. The third, the continuity equation, describes the conservation of mass within the system. These equations take the form:

(B1) $$\frac{D\boldsymbol{v}}{Dt} + f\hat{k} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p$$

(B2) $$\frac{\partial p}{\partial z} = -\rho g$$

(B3) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

where *D*/*Dt* is the total derivative, boldface variables denote vectors, and all variables adhere to typical meteorological conventions.

There are many approximations inherent to the shallow-water equation system, some of which are reflected in the above equations and others of which are implicit:

- The concept of "shallow water" means that we assume that the vertical scale is much smaller than the horizontal scale. For equatorial waves, this is a fair assumption: vertical wavelengths are on the order of 15-40 km whereas horizontal wavelengths are on the order of 1,000 km or more. Further, the medium along which the waves propagate is even smaller at tens to hundreds of meters wide.
- The atmosphere can be approximated by two layers with constant densities (ρ_1 in the lower layer, ρ_2 in the upper layer). The interface between the lower and upper layers is the medium along which a wave propagates. We assume that the atmosphere is stable stratified, such that the lower layer's density is greater than that of the upper layer (i.e., $\rho_1 > \rho_2$, such that $p_1 > p_2$).
- Since we invoked the hydrostatic approximation, the horizontal pressure gradient in both layers is independent of height. This can be demonstrated by taking a horizontal derivative (e.g., with respect to *x*) of (B2) and commuting the order of the partial derivatives for *p*.
- Waves in the shallow-water system have finite amplitudes (i.e., are quasi-linear in nature).
- The equations are incompressible, with density conserved following the motion, thus sound waves cannot be possible solutions.
- Friction is neglected.
- Finally, we assume that there is no horizontal pressure gradient in the *upper* layer.

To simplify our set of equations above, particularly (B1), we desire an expression for the horizontal pressure gradient in the *lower* layer. In other words, we want to know how pressure varies between points B and A within the model depicted in the "Equatorial Waves" lecture materials. Since there is no horizontal pressure gradient in the upper layer, the pressure along the interface between the layers (above point B) is equivalent to that within the upper layer (above point A). We first assume that differences in pressure along and near the interface between the upper and lower layers are infinitesimally small. The pressure at point A is thus a function of a pressure displacement associated with a downward-forced upper layer and the pressure at point B is thus a function of a pressure displacement associated with an upward-forced lower layer. We will refer to these displacements as δp_2 and δp_1, respectively. Thus, the pressure at points A and B is given by:

(B4a) $$A\colon p+\delta p_2$$

(B4b) $$B\colon p+\delta p_1$$

We can use (B2) to re-write (B4) in terms of the density within each layer and the displacement in height associated with the wave, such that:

(B5a) $$p+\delta p_2 = p+\rho_2 g\delta z$$

(B5b) $$p+\delta p_1 = p+\rho_1 g\delta z$$

It should be noted in (B5) that the leading negative associated with the hydrostatic approximation is folded into the vertical displacement variable ($\partial z = z_1 - z_2$).

Next, let the height of the interface at the point above point B be equal to h_2. Similarly, let the height of the interface at point A be equal to h_1. In this case, δz is merely equal to $h_2 - h_1$. However, let us consider the case where the distance between points B and A, δx, is infinitesimally small ($\delta x \approx 0$). In this case,

(B6) $$\delta z = \frac{h_2-h_1}{\partial x}\delta x = \frac{\partial h}{\partial x}\delta x$$

If we substitute (B6) into (B5), divide by δx, and subtract the expression for point A (B5a) from that for point B (B5b), we obtain the following:

(B7) $$\lim_{\delta x\to 0}\left[\frac{(p+\delta p_1)-(p+\delta p_2)}{\delta x}\right] = g\delta\rho\frac{\partial h}{\partial x}$$

where $\delta\rho$ is equal to ρ_1-ρ_2. The limit notation arises because of the assumption that $\delta x \approx 0$. The corresponding expressions for the meridional pressure (and height) gradients can be obtained in a similar manner and take identical forms (except in terms of y). These expressions are akin to saying that horizontal pressure gradients are equivalent to horizontal fluid-depth gradients.

If we substitute the right-hand side of (B7) and accompanying expression for $\partial h/\partial y$ into (B1) and expand into the full u-momentum and v-momentum equations, we obtain:

(B8a) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = fv - g\frac{\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

(B8b) $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -fu - g\frac{\delta\rho}{\rho_1}\frac{\partial h}{\partial y}$$

Next, express the fluid depth h, which is a function of (x,y,t), in terms of a constant height plus a perturbation height, i.e., (B9) $$h(x,y,t) = H + h'(x,y,t)$$

In the above, *H* is defined as the equivalent depth and is proportional to the stability. It impacts the vertical wavenumber and thus the wave's vertical structure and depth. In the tropics, *H* generally ranges between 10-500 m (smaller for dry dynamics, larger for moist dynamics) for large vertical wavelengths (5-50 km). Thus, equatorial-wave fluid depths can be large, facilitating vertical wave structure throughout the fluid!

If we substitute (B9) into (B8), the constant term drops out (since it is constant in *x* and *y*), such that:

(B10a) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}$$

(B10b) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial y}$$

These represent our shallow-water horizontal momentum equations.

Next, we wish to manipulate the continuity equation (B3). We wish to integrate both sides of (B3) from the ground, where the vertical motion must be equal to zero, to *h*, the fluid height. Since the pressure gradient expressed in (B7) is independent of height (i.e., not a function of *z*), (B10) makes it clear that both *u* and *v* are independent of height (and thus are not functions of *z*) presuming that they were not functions of height at the initial time $t = 0$. As a result, our integration is simplified. First, the left-hand side of (B3):

(B11a) $$\int_{z'=0}^{z'=h}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz' = h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Next, the right-hand side of (B3):

(B11b) $$-\int_{z'=0}^{z'=h}\left(\frac{\partial w}{\partial z}\right)dz' = -[w(z'=h) - w(z'=0)]$$

Since our lower boundary is flat and rigid, $w(z' = 0)$ equals 0. Meanwhile, *w* at the fluid interface is merely a reflection of the vertical movement of the fluid-interface itself and can be expressed as:

(B12) $$w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

If we simplify (B11b), substitute (B12) into (B11b), and set the result equal to (B11a), we obtain:

(B13) $$-h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

In the process of obtaining (B13), we multiplied both sides by -1 to bring the leading negative on (B11b) over to the left-hand side of the continuity equation.

Finally, substitute (B9) into (B13) to obtain:

(B14) $$-(H+h')\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial h'}{\partial t}+u\frac{\partial h'}{\partial x}+v\frac{\partial h'}{\partial y}$$

This continuity equation states that fluid-interface motion (given by the right-hand side of (B14), manifest only in the perturbation height) equals the depth of the lower fluid times the convergence in the lower fluid. Either greater or deeper convergence will lead to greater fluid-interface motion.

Next, we wish to simplify the Coriolis term in (B10). We introduce the concept of a Beta plane, where the range of latitudes being considered is sufficiently small such that the meridional variation in the Coriolis parameter can be treated as a linear rather than non-linear (e.g., sin ϕ) function of y. This approximation results from performing a Taylor series expansion on f and keeping only the first two terms, such that:

(B15) $$f=f_0+\beta y, \beta=\frac{\partial f}{\partial y} \text{ (assuming } y = 0 \text{ at the equator)}$$

Recalling that $f = 2\Omega\sin\phi$, where ϕ is latitude, β thus equals $(2\Omega\cos\phi)/a$, where a is the Earth's radius and is generally assumed to have a constant value of 6.37 x 10^6 m. If we employ the small angle approximation (i.e., ϕ small), then $\cos\phi \approx 1$ and $\beta = 2\Omega/a$ = 2.3 x 10^{-11} m^{-1} s^{-1}.

If we restrict (B15) to the Equator, where f_0 is zero, then $f = \beta y$. Substitute this into (B10) and (B14) to obtain:

(B16a) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta yv-g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}$$

(B16b) $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\beta yu-g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial y}$$

(B17) $$-(H+h')\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial h'}{\partial t}+u\frac{\partial h'}{\partial x}+v\frac{\partial h'}{\partial y}$$

where (B17) is identical to (B14) because there are no Coriolis terms in (B14).

Finally, we linearize (B16) and (B17) about a background state that is initially at rest. In this process, we assume that the three variables represented within the system given by (B16) and (B17) – u, v, and h – can be partitioned into mean (or background flow) and perturbation (or wave flow) components:

(B18a) $$u(x,y,t)=\bar{u}(x,y,t)+u'(x,y,t)$$

(B18b) $$v(x,y,t)=\bar{v}(x,y,t)+v'(x,y,t)$$

(B18c) $$h(x,y,t)=H+\bar{h}(x,y,t)+h'(x,y,t)$$

Equation (B18c) has three terms because our initial definition of h (given by (B9)) did not allow for spatial or temporal variance in the base-state H. The perturbation fields u', v', and h' inherently must be small. The mean terms in (B18) are zero if we assume a background state lacking horizontal or vertical motion, which we will do here.

Note that equatorial waves in the shallow-water system are expressed here in terms of three dimensions: *x*, *y*, and *t*. However, it is possible to formulate a vertical structure equation that describes the vertical structure of these waves. This equation is a second-order partial differential equation and is a function of the vertical wavenumber *m*, itself largely dependent upon stability (e.g., as enters through the equivalent depth *H*). As noted above, the vertical wavelength of these equatorial waves is typically on the order of 25-40 km. The vertical structure equation can be used to provide detail of a wave's structure over that wavelength. For our purposes, however, it is most important to know that equatorial waves do contain vertical structure.

If we substitute (B18) into (B16) and (B17) and simplify, we obtain:

(B19a) $$\frac{\partial u'}{\partial t} - \beta y v' = -g' \frac{\partial h'}{\partial x}$$

(B19b) $$\frac{\partial v'}{\partial t} + \beta y u' = -g' \frac{\partial h'}{\partial y}$$

(B20) $$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

where we have moved the β terms to the left-hand side of the equations. We have also substituted *effective gravity g'* into the right-hand side of (B16), as defined by:

(B21) $$g' = g\frac{\delta\rho}{\rho_1}$$

Terms involving products of linearized variables – namely, the advection terms – in (B16) and (B17) vanish and are thus not present in (B19) and (B20). Specifically, substituting (B18) into the advection terms results in four terms. Three of the terms involve the mean fields in some way. Since we said above that these terms are zero, the three mean terms vanish. The remaining term involves the product of two perturbation fields. As perturbations are inherently small, their product is assumed to be negligible, causing this term to vanish.

We are now ready to explore solutions to (B19) and (B20). These equations have three unknown variables: *u'*, *v'*, and *h'*. We assume wave-like solutions for each of these three variables as follows:

(B23a) $$u'(x, y, t) = U(y) * exp(\, i(kx - \omega t))$$

(B23b) $$v'(x, y, t) = V(y) * exp(\, i(kx - \omega t))$$

(B23c) $$h'(x, y, t) = H_w(y) * exp(\, i(kx - \omega t))$$

The functions *U*, *V*, and H_w describe each wave's amplitudes, which are said to vary only in the meridional (*y*) direction. In the above, *k* is the zonal wavenumber, ω is the wave's frequency (equal to the number of times the wave passes a given point per second, and thus related to its propagation), and *i* equals the square root of -1.

Before substituting (B23) into (B19) and (B20), note that derivatives of (B23) with respect to *x* and *t* have special forms given by:

(B24a) $$\frac{\partial}{\partial x}(\quad) = ik(\quad)$$

(B24b) $$\frac{\partial}{\partial t}(\quad) = -i\omega(\quad)$$

These arise because $U(y)$, $V(y)$, and $H_w(y)$ are not functions of x or t and because the derivative of an exponential function is equal to the derivative of the exponential multiplied by the exponential function.

Substituting (B23) into (B19) and (B20), making use of the definitions in (B24), and dividing through by a common factor of the exponential wave function $exp(\, i(kx - \omega t))$, we obtain:

(B25a) $$-i\omega U - \beta y V = -ikg' H_W$$

(B25b) $$-i\omega V + \beta y U = -g' \frac{\partial H_W}{\partial y}$$

(B26) $$-i\omega H_w + H\left(ikU + \frac{\partial V}{\partial y}\right) = 0$$

where we have dropped the functional (y) notation on U, V, and H_W for simplicity.

The three equations given by (B25) and (B26) have three unknown variables: U, V, and H_w. We now wish to simplify this system into one of two equations for two variables, after which we will simplify further into a single equation for a single variable. First, we solve (B25a) for U:

(B27) $$U = -\frac{\beta y V}{i\omega} + \frac{kg' H_W}{\omega}$$

We then substitute (B27) into (B25b), multiply all terms by -$i\omega$, group like terms (specifically, the V terms), and rearrange slightly to obtain:

(B28) $$(B^2 y^2 - \omega^2)V - ikg'\beta y H_W - i\omega g' \frac{\partial H_W}{\partial y} = 0$$

Similarly, we then substitute (B27) into (B26), multiply all terms by $i\omega$, group the like V and H_w terms, and rearrange slightly to obtain:

(B29) $$(\omega^2 - Hk^2 g')H_W + iH\omega(\frac{\partial V}{\partial y} - \frac{k\beta y}{\omega} V) = 0$$

We follow a similar procedure to reduce (B28) and (B29) into a single equation by first solving (B29) for H_w and substituting the result into (B28) to obtain a single equation for V.

Note that from (B29), H_w and its derivative with respect to y are given by the following:

(B30a) $$H_W = \frac{-i\omega H \frac{\partial V}{\partial y} + ikH\beta y V}{\omega^2 - g' H k^2}$$

(B30b) $$\frac{\partial H_W}{\partial y} = \frac{-i\omega H\frac{\partial^2 V}{\partial y^2} + ikH\beta y\frac{\partial V}{\partial y} + ikH\beta V}{\omega^2 - g'Hk^2}$$

where we used the product rule to obtain (B30b) from (B30a). Substituting (B30) into (B28) gives a second-order partial differential equation for *V* with three sets of terms: *V*, its first derivative with respect to *y*, and its second derivative with respect to *y*. Specifically, these terms take the form:

(B31a) $$V\left[(\beta^2 y^2 - \omega^2) + \frac{k\omega g'h\beta}{\omega^2 - Hk^2 g'} + \frac{k^2 g'\beta^2 y^2 H}{\omega^2 - Hk^2 g'}\right]$$

(B31b) $$\frac{\partial V}{\partial y}\left[\frac{-kg'H\omega\beta y}{\omega^2 - Hk^2 g'} + \frac{kg'H\omega\beta y}{\omega^2 - Hk^2 g'}\right]$$

(B31c) $$\frac{\partial^2 V}{\partial y^2}\left[\frac{-\omega^2 g'H}{\omega^2 - Hk^2 g'}\right]$$

the sum of which equals zero. Note that (B31b) alone equals zero since the terms inside the brackets cancel each other out.

If we multiply (B31a) by $\omega^2 - Hk^2 g'$, simplify the result, divide by $-\omega^2 g'H$, and rearrange, the final result – a second-order partial differential equation for *V* – is obtained, given by:

(B32) $$\frac{\partial^2 V}{\partial y^2} + \left(\frac{\omega^2}{g'H} - \frac{\beta^2 y^2}{g'H} - k^2 - \frac{\beta k}{\omega}\right)V = 0$$

If solution(s) for *V* are obtained from (B32), they can be used with (B30a) to obtain solution(s) for H_w and, subsequently, with (B27) to obtain solution(s) for *U*.

We require solutions to (B32) where *V* approaches 0 as the distance from the Equator *y* grows increasingly large (i.e., approaches ±∞). This constrains our solutions to have maximum amplitude near the equator and to decay north and south from there. In his seminal work on equatorial waves, Matsuno (1966) demonstrated that the solutions for (B32) only satisfy this condition if there is a finite odd integer number of waves present in the meridional direction, i.e.,

(B33a) $$\frac{\sqrt{g'H}}{\beta}\left(\frac{\omega^2}{g'H} - k^2 - \frac{\beta k}{\omega}\right) = 2n + 1$$

Or, equivalently,

(B33b) $$\frac{\omega^2}{g'H} - \frac{\beta}{\sqrt{g'H}} - k^2 - \frac{\beta k}{\omega} = \frac{2n\beta}{\sqrt{g'H}}$$

(B33) is obtained from by considering the coefficient on *V* in (B32) in the context of *y* approaching ±∞ and how it modulates the solution to the second-order partial differential equation given by (B32). The value *n* is a general wavenumber and is equal to 0, 1, 2, 3, and so on.

Equation (B33a) relates the frequency ω and zonal wavenumber k for all possible wave solutions (n = 0, 1, 2, 3…) and thus provides the basis for the generic *dispersion relation* of the system. The dispersion relation is cubic, i.e., dependent upon ω^3 (if multiplied through by ω to eliminate the $1/\omega$ term):

(B34)
$$\frac{\omega^3}{g'H} - \omega\left(\frac{\beta}{\sqrt{g'H}} - k^2 - \frac{2n\beta}{\sqrt{g'H}}\right) = \beta k$$

such that there are at most three unique solutions to (B34). These solutions are associated with three of the four equatorial waves: equatorial Rossby ($n \geq 1$), mixed Rossby-gravity ($n = 0$), and inertia-gravity waves ($n \geq 1$). As Kelvin waves have no meridional structure, these solutions in V do not directly describe Kelvin waves; instead, they are accounted for separately, as will be demonstrated in subsequent sections. However, it should be noted that Kelvin waves can also be obtained from (B34) for the special case where $n = -1$.

The solutions described here are obtained in the absence of diabatic heating and thus are unforced solutions. When a generic diabatic heating is included, the most-common solution is a combination of the equatorial Rossby and Kelvin waves, representing the waves with the smallest frequencies. The lower-frequency wave modes dominate this forced solution because these modes are more responsive to the heat forcing than are higher-frequency wave modes (Matsuno 1966).

From (B34), we see that there are two primary physical forcings on V. The first is buoyancy, analogous to $g'H$ and thus akin to potential energy, wherein the equivalent depth H is a function of the divergence in the lower layer. The second is β. The solution's structure is modulated by the frequency ω, zonal wavenumber k, and the distance along the meridional axis y.

Solutions for V in (B32) have the general form of solutions to the Schrodinger equation for an oscillatory wave mode, given by:

(B35)
$$V(Y) = A\,exp(-\frac{Y^2}{2})H_N(Y)$$

In (B35), A is an amplitude function, $H_N(Y)$ are Hermite polynomials of order N and are integers and/or some multiple or power of Y, and Y is defined by:

(B36)
$$Y = \left(\frac{\sqrt{g'H}}{\beta}\right)^{1/2} y$$

Note that (B36) is slightly different than that contained within Appendix C of Chapter 4 in *An Introduction to Tropical Meteorology*. Y is proportional to the ratio of buoyancy to β. Solutions for V can be found for individual values of N. These solutions, as noted above, can be used with (B30a) to obtain solutions for H_w and, subsequently, with (B27) to obtain solutions for U. These give the meridional amplitudes of solutions for u', v', and h', each with wave-like structure in x and t (and a vertical structure that is unspecified herein).

As an aside, the Rossby radius of deformation L_R (the horizontal distance at which buoyancy effects, which primarily govern the smaller-scale flow, have equal importance to rotational effects, which primarily govern the larger-scale flow, in describing the flow's evolution) is typically on the order of 6,000 km in the tropics. As described above, however, equatorial waves have horizontal length scales on the order of 1,000-2,000 km. Equatorial waves thus have smaller horizontal length scales than the Rossby radius of deformation and are weakly constrained by rotation. In the context of geostrophic adjustment, this means that the mass fields such as pressure and temperature adjust more significantly than do the wind fields to achieve a geostrophic balance. This results in broad, relatively weak atmospheric perturbations with equatorial waves, as expected given the assumptions of linearity and finite amplitudes underpinning this derivation. Indeed, the magnitude of the wind and pressure perturbations associated with each of the equatorial wave modes is relatively small (e.g., 1-2 m s^{-1} or hPa, of similar magnitudes to routine observational uncertainties in each quantity).

As noted earlier, Kelvin waves have no meridional motion. Thus, (B33) is not exactly valid for these waves and we need another way of deriving their dispersion relation. Returning to (B19) and (B20), if we set v' = 0 in these equations to explicitly prohibit meridional motions, we obtain:

(B37a) $$\frac{\partial u'}{\partial t} = -g'\frac{\partial h'}{\partial x}$$

(B37b) $$\beta y u' = -g'\frac{\partial h'}{\partial y}$$

(B38) $$\frac{\partial h'}{\partial t} + H\frac{\partial u'}{\partial x} = 0$$

These equations give us a set of three equations for u' and h'. If we substitute the wave-like solutions for u' and h' given by (B23a) and (B23c) into (B38) and use (B24) to simplify the result, we obtain:

(B39) $$-i\omega H_w + HikU = 0$$

Solving this for U, we obtain:

(B40) $$U = \frac{\omega H_w}{kH}$$

If we substitute (B40), (B23a), and (B23c) into (B37a), we obtain:

(B41) $$\frac{i\omega^2 H_w}{kH} = g'ikH_w$$

Finally, solving (B41) for ω, we obtain:

(B42) $$\omega = \pm k\sqrt{g'H}$$

This is identical to the dispersion relation of a pure gravity wave and can be obtained for small n and large k from the inertia-gravity wave's dispersion relation, such that the equatorial Kelvin wave can be viewed as a special case of a gravity wave.