Appendix D: Derivation of the Sawyer-Eliassen Equation for the Tropical Cyclone Secondary Circulation

Below are the fundamental equations represented in a two-dimensional (z, r) cylindrical coordinate system, in which the cyclone is treated as axisymmetric (having uniform structure in all directions at a given radius):

(D1)
$$m^2 = r^3 \frac{\partial \varphi}{\partial r}$$

(D2)
$$\frac{dm^2}{dt} = F$$

(D3)
$$\frac{\partial \Phi}{dz} = \frac{g}{\theta_0} \theta$$

(D4)
$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$

(D5)
$$\frac{d\theta}{dt} = Q$$

(D6)
$$\Phi = \varphi + \frac{f^2 r^2}{8}$$

(D7)
$$z = \left[1 - \left(\frac{\rho}{\rho_0}\right)^K\right] \frac{c_p \theta_0}{g}$$

(D8)
$$\frac{1}{r^3} \frac{\partial m^2}{\partial z} = \frac{g}{\theta} \frac{\partial \theta}{\partial r}$$

Except as noted below, all variables have their standard meaning. Subscripts of 0 denote base-state values. From top to bottom, these equations represent:

- (D1): Gradient-wind balance.
- (D2): As angular momentum is a function of the tangential wind v, (D2) is the tangential momentum equation. Changes in angular momentum following the motion are exclusively related to prescribed momentum forcing F.
- (D3): The hydrostatic equation.
- (D4): The flux form of the continuity equation. Recall that u denotes radial and not zonal wind.
- (D5): The thermodynamic equation, indicating that changes in potential temperature following the motion are exclusively related to diabatic heating at a rate given by Q.
- (D6): Definition of the geopotential height.
- (D7): Definition of a pseudoheight vertical coordinate to simplify the math. Note that the exponent K is equal to R_d/c_p.
- (D8): Thermal-wind balance, relating the vertical wind shear (angular momentum) to horizontal potential-temperature gradients. We assume a tropical cyclone initially in thermal-wind balance in the absence of any prescribed heat (Q) or momentum (F) forcings. Prescribing one or both of these

forcings destroys thermal-wind balance, and the Sawyer-Eliassen equation that we are deriving will show how the tropical cyclone's secondary circulation responds to restore thermal-wind balance.

First, expand the total derivatives in (D2) and (D5) to obtain:

(D9)
$$\frac{\partial m^2}{\partial t} + u\frac{\partial m^2}{\partial r} + w\frac{\partial m^2}{\partial z} = F$$

(D10)
$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z} = Q$$

Next, take the partial derivative of (D9) with respect to z and multiply the result by $1/r^3$:

(D11)
$$\frac{\partial}{\partial t}\left(\frac{1}{r^3}\frac{\partial m^2}{\partial z}\right) + \frac{\partial}{\partial z}\left(u\frac{1}{r^3}\frac{\partial m^2}{\partial r} + w\frac{1}{r^3}\frac{\partial m^2}{\partial z}\right) = \frac{1}{r^3}\frac{\partial F}{\partial z}$$

Similarly, take the partial derivative of (D10) with respect to r and multiply the result by g/θ_0:

(D12)
$$\frac{\partial}{\partial t}\left(\frac{g}{\theta_0}\frac{\partial \theta}{\partial r}\right) + \frac{\partial}{\partial r}\left(u\frac{g}{\theta_0}\frac{\partial \theta}{\partial r} + w\frac{g}{\theta_0}\frac{\partial \theta}{\partial z}\right) = \frac{g}{\theta_0}\frac{\partial Q}{\partial r}$$

In obtaining (D11) and (D12), partial derivatives with respect to time have been commuted with the partial derivatives with respect to z and r, respectively.

Before proceeding, it is useful to define several additional terms to simplify (D11) and (D12):

(D13)
$$N^2 = \frac{g}{\theta_0}\frac{\partial \theta}{\partial z}$$

(D14)
$$B = -\frac{g}{\theta_0}\frac{\partial \theta}{\partial r} = -\frac{1}{r^3}\frac{\partial m^2}{\partial z}$$

(D15)
$$I = \frac{1}{r^3}\frac{\partial m^2}{\partial r} = \left(f + \frac{1}{r}\frac{\partial(rv)}{\partial r}\right)\left(f + \frac{2v}{r}\right)$$

(D13-15) define static stability, baroclinicity, and inertial stability, respectively. Applying these definitions to (D11) and (D12) results in the following:

(D16)
$$\frac{\partial}{\partial t}(-B) + \frac{\partial}{\partial z}(Iu - Bw) = \frac{1}{r^3}\frac{\partial F}{\partial z}$$

(D17)
$$\frac{\partial}{\partial t}(-B) + \frac{\partial}{\partial r}(-Bu + N^2 w) = \frac{g}{\theta_0}\frac{\partial Q}{\partial r}$$

Next, subtract (D16) from (D17) to eliminate the time partial derivatives and obtain:

(D18)
$$\frac{\partial}{\partial r}(N^2 w - Bu) + \frac{\partial}{\partial z}(Bw - Iu) = \frac{g}{\theta_0}\frac{\partial Q}{\partial r} - \frac{1}{r^3}\frac{\partial F}{\partial z}$$

(D18) describes the zonal (u) and vertical (w) velocity responses to imposed heat (Q) and momentum (F) forcing. However, because there are two unlinked unknowns given by u and w, this equation is difficult to solve. To link these two variables and thus make solving the diagnostic equation simpler, the streamfunction (describing motion in the radius-vertical, or r-z, plane) is used.

The streamfunction in this coordinate system is defined by:

(D19)
$$u = -\frac{\partial \psi}{\partial z}, w = \frac{1}{r}\left(\frac{\partial (r\psi)}{\partial r}\right)$$

Substituting (D19) into (D18) results in the following:

(D20)
$$\frac{\partial}{\partial r}\left(N^2 \frac{1}{r}\frac{\partial (r\psi)}{\partial r} + B\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(B\frac{1}{r}\frac{\partial (r\psi)}{\partial r} + I\frac{\partial \psi}{\partial z}\right) = \frac{g}{\theta_0}\frac{\partial Q}{\partial r} - \frac{1}{r^3}\frac{\partial F}{\partial z}$$

(D20) is the Sawyer-Eliassen non-linear secondary-circulation diagnostic equation, relating streamfunction ψ to imposed heating Q and momentum F sources and modulated by static stability N^2, inertial stability I, and baroclinicity B. *The streamfunction responds to attempt to restore the thermal wind balance that the specified heating and/or momentum forcing disrupts!* While thermal wind balance restoration is never truly achieved, the concepts of balance destruction and restoration nevertheless enable us to consider how radial and vertical motions are impacted by prescribed heating and/or momentum forcing.