Equatorial Waves

Introduction

Given specific assumptions, we can obtain solutions to simplified dynamical equations that represent a set of equatorially trapped, approximately geostrophic tropical waves known as *equatorial waves*. These waves can be unforced or forced by localized diabatic heating. This lecture introduces the most-common equatorial wave types and their observational characteristics, considers their basic dynamics, and describes how they can be identified and tracked within routinely available satellite and model data.

Key Questions

- How do the kinematic structures, pressure anomalies, propagation speeds and directions, horizontal scales, and restoring mechanisms differ between Kelvin, equatorial Rossby, inertia-gravity, and mixed Rossby-gravity waves?
- What effect does deep, moist convection universally have on these waves' propagation speeds?
- How can each waves' connection to deep, moist convection be used to identify and track equatorial waves in real-time using satellite data?

An Introduction to Equatorial Waves

There are four primary types of equatorial waves that we are concerned with in this course. These are the *Kelvin*, *equatorial Rossby*, *mixed Rossby-gravity*, and, to lesser extent, *inertia-gravity* waves. Each of these waves represent specific solutions to the shallow-water equation system, a simplified equation set that can be used to describe atmospheric phenomena. Equatorial waves can be unforced (dry) or forced by diabatic heating (moist or convectively coupled). Latent heat release in deep, moist convection (convective heating) is the most common diabatic driver of equatorial waves. Herein, we describe each wave type, consider basic dynamical properties of each wave type, and then discuss how we can identify and track these waves.

An Introduction to the Predominant Equatorial Wave Modes

Kelvin Waves

Kelvin waves are large-scale waves that propagate along a physical boundary such as a mountain range or coastline. In the tropics, the Equator, where $f = 0$, is a trapping barrier between the Northern and Southern Hemispheres, where $f \neq 0$. Thus, equatorial Kelvin waves are said to be "equatorially trapped" waves. This barrier results in Kelvin waves having no meridional velocity; the horizontal flow is purely zonal, or east-to-west, in nature. The Coriolis force, specifically it being non-zero away from the Equator, and buoyancy are a Kelvin wave's *restoring mechanisms*, or the forces that cause a feature to oscillate like a wave.

Kelvin waves have a return period of ~6-7 days. As a result, they are a leading cause of short-term variations in several phenomena. For instance, Kelvin waves exert a significant influence on deep, moist convection within 10° latitude of the Equator. This impact is typically greatest in the eastern Indian and central Pacific Oceans, with the magnitude of this impact over Africa, South America, and the western Indian Ocean being somewhat variable with the seasons. Furthermore, convectively coupled Kelvin waves (indicating Kelvin waves that occur in association deep, moist convection) facilitate short-term variations in the zonal Walker circulation, the characteristics of which we will discuss in a later lecture. Finally, the westerly wind bursts that often accompany Kelvin waves can be a harbinger of El Niño development and, separately, can increase the likelihood of tropical cyclone formation in the tropics.

Kelvin waves have a horizontal length scale of approximately 2,000 km, and it is diabatic heating centered on the Equator with this length scale that is the most common forcing mechanism for Kelvin wave initiation. Convectively coupled Kelvin waves, representing Kelvin waves associated with and linked to deep, moist convection, propagate eastward with a phase speed of 12-25 m s^{-1}. Dry, unforced Kelvin waves propagate eastward at a somewhat faster phase speed. In general, for all types of equatorial waves (not just for Kelvin waves), deep, moist convection results in a slowing of the wave's propagation speed.

Equatorial Rossby Waves

Midlatitude Rossby waves result from meridional potential-vorticity gradients. Their equatorial equivalents result from, and have as their restoring mechanism, meridional planetary-vorticity gradients. These waves are associated with twin vortices on either side of the Equator; such vortices occur most often in the Indian Ocean and western Pacific Ocean. The direct impacts of equatorial Rossby waves are strongest over Asia and the western Pacific Ocean. These waves have a horizontal length scale of approximately 1,000 km, and they are commonly forced by diabatic heating that is *symmetric* across the Equator. Their duration is on the order of several days. Convectively coupled equatorial Rossby waves move westward with a phase speed between 5-7 m s^{-1}, whereas their dry counterparts move westward with a phase speed between 10-20 m s^{-1}.

Mixed Rossby-Gravity Waves

Mixed Rossby-gravity waves are forced by and subsequently initiate deep, moist convection. These waves have characteristics of both inertia-gravity waves (which are tied to buoyancy) and equatorial Rossby waves (which are tied to meridional planetary-vorticity gradients). As a result, the restoring mechanisms for mixed Rossby-gravity waves are buoyancy and the meridional variation of the Coriolis parameter. Mixed Rossby-gravity waves are typically tilted northwest-to-southeast across the Equator by the convective asymmetries that often accompany them. Mixed Rossby-gravity waves are most common across the equatorial western and central Pacific Ocean, particularly during Northern Hemisphere summer and autumn. The horizontal length scale of mixed Rossby-gravity waves is approximately 1,000 km, similar to that of equatorial Rossby waves. However, unlike equatorial Rossby waves, mixed Rossby-gravity are forced by diabatic heating that is *asymmetric* across the Equator. They have a period of 4-5 days and move westward at a phase speed of approximately 8-10 m s^{-1}.

Physical Description of the Equatorial Wave Solutions

While equatorial waves exist within the real atmosphere, it is easier to consider their structures in the context of a simplified atmospheric system. The shallow-water equations represent such a simplified atmospheric system. The shallow-water equations are applicable when the horizontal length scale far exceeds the vertical length scale, as is true for equatorial waves (with horizontal length scales of ~1,000 km and vertical length scales of ~40-50 km).

A derivation of the equatorial-wave solutions for the shallow-water equations is given in Appendix B. There are three variables of interest within the shallow-water equation system: the zonal velocity *u*, the meridional velocity *v*, and the fluid depth *h*, the latter of which is analogous to pressure *p*. We prescribe linear solutions for *u*, *v*, and *h* by linearizing them, wherein each variable is said to be the sum of a mean state (assumed to be at rest, or zero; e.g., $\bar{u}$) and a perturbation component uniquely associated with the wave (e.g., *u'*).

These variables have wave-like solutions of the general form:

(1a) $$u'(x, y, t) = U(y) * exp(\, i(kx - \omega t))$$

(1b) $$v'(x, y, t) = V(y) * exp(\, i(kx - \omega t))$$

(1c) $$h'(x, y, t) = H_w(y) * exp(\, i(kx - \omega t))$$

The exponential functions give each variable their wave-like structures; this can be shown by using Euler's relations to rewrite the exponentials in terms of sine and cosine waves. These waves vary in the horizontal (*x*, *y*) and temporal (*t*) dimensions, and we typically neglect their vertical structure for simplicity. The *U*(*y*), *V*(*y*), and $H_w(y)$ functions characterize the waves' amplitudes, which only vary in the meridional direction and are prescribed to decay to zero away from the Equator. In (1a-c), *k* is the zonal wavenumber, inversely proportional to the zonal wavelength, and ω is the frequency, equal to the number of times the wave passes a given point per second and thus related to its propagation velocity.

The seminal work of Matsuno (1966) shows that the wave-like solutions defined in (1a-c) decay toward 0 away from the Equator only if the following relationship for the frequency ω holds:

(2) $$\frac{\sqrt{g'H}}{\beta}\left(\frac{\omega^2}{g'H} - k^2 - \frac{\beta k}{\omega}\right) = 2n + 1$$

where, *g'H* is a measure of buoyancy, β is the meridional variation in the Coriolis parameter, and the value *n* is a generic integer wavenumber. Eqn. (2) relates the frequency ω and zonal wavenumber *k* for all possible wave solutions (n = 0, 1, 2, 3…). In dynamical parlance, it represents the generic *dispersion relation* of the equatorial-wave solutions. It can be used to define the *phase speed* c_p (wave propagation) and *group velocity c*g (energy propagation), where:

$$c_p = \frac{\omega}{k} \qquad\qquad c_g = \frac{\partial\omega}{\partial k}$$

Because (2) is a cubic equation for ω, it has at most three unique solutions. These solutions are associated with three of the four equatorial waves: equatorial Rossby ($n \geq 1$), mixed Rossby-gravity ($n = 0$), and inertia-gravity waves ($n \geq 1$). Because Kelvin waves have no meridional structure, these solutions do not directly describe Kelvin waves; instead, they are accounted for separately. However, it should be noted that Kelvin waves can also be obtained from (2) for the special case where $n = -1$.

We now turn to examining the unique solutions and dispersion relations for each equatorial wave mode.

Equatorial Rossby Waves

Equatorial Rossby waves have the longest period or duration of the equatorial waves at ~14-21 days. Since a wave's frequency is inversely related to its period, equatorial Rossby waves have the lowest frequency of the equatorial waves. This implies that ω is small and allows us to neglect the ω^2 term in (2), such that:

(3) $$-\frac{\sqrt{g'H}}{\beta}(k^2 + \frac{\beta k}{\omega}) = 2n + 1$$

If we rearrange (3) to solve for ω, we obtain:

(4) $$\omega = -\frac{\beta k}{\left(k^2 + \frac{\beta(2n+1)}{\sqrt{g'H}}\right)}$$

And, consequently, the phase speed c_p is given by:

(5) $$c_p = -\frac{\beta}{\left(k^2 + \frac{\beta(2n+1)}{\sqrt{g'H}}\right)}$$

Because each of the variables in (5) are positive-definite (β is positive since the Coriolis parameter increases to the north, or along the positive *y*-axis), $c_p < 0$, such that equatorial Rossby waves propagate *westward*. Their propagation speed is determined chiefly by buoyancy and the meridional planetary vorticity gradient, with lesser contributions from the zonal wavenumber k (inversely related to the zonal wavelength) and the generic wavenumber n.

The theoretical solution for $n = 1$ equatorial Rossby waves is characterized by westward-propagating high- and low-pressure centers mirrored across the Equator. Horizontal velocity is maximized along the Equator. Deep, moist convection is preferentially found where convergence is maximized: east of low pressure and west of high pressures. Note that there is also confluence near the Equator west of low pressure and east of high pressure. This confluence is weak and is largely mitigated by speed divergence, however. Deep, moist convection redistributes positive potential vorticity from higher to lower altitudes, increasing cyclonic flow east of areas of low pressure. This tugs at the wave, slowing its westward propagation.

Inertia-Gravity Waves

These buoyancy-dependent waves have high frequency (large ω) and short longevity. As a result, the $-\beta k/\omega$ term in (2) is negligibly small. This enables us to express the dispersion relation as the following:

(6) $$\frac{\sqrt{g'H}}{\beta}(\frac{\omega^2}{g'H} - k^2) = 2n + 1$$

Solving for ω in (6) by re-arranging terms, multiplying both sides by $\beta\sqrt{g'H}$, and taking the square root of the result, we obtain:

(7) $$\omega = \pm\sqrt{g'Hk^2 + \sqrt{g'H}\beta(2n+1)}$$

With the phase speed given by:

(8) $$c_p = \pm \frac{\sqrt{g'Hk^2 + \sqrt{g'H}\beta(2n+1)}}{k}$$

From (8), we see that inertia-gravity waves have two modes of propagation, one each to the east and west, that correspond to the phase speed's positive and negative roots.

Mixed Rossby-Gravity Waves

For the special case of mixed Rossby-gravity waves, we let $n = 0$ in (2) such that:

(9) $$(\frac{\omega^2}{g'H} - k^2 - \frac{\beta k}{\omega}) = \frac{\beta}{\sqrt{g'H}}$$

Or, expressed in terms of ω^3,

(10) $$\frac{\omega^3}{g'H} - \omega\left(k^2 + \frac{\beta}{\sqrt{g'H}}\right) - \beta k = 0$$

There are three possible roots to the cubic equation given by (10). It is possible to solve for these using mathematical techniques designed to solve cubic equations; however, as this is a tedious, time-consuming process, we will instead focus on the solutions themselves. Two of these solutions (or roots) are akin to the inertia-gravity wave roots described by (7). However, the westward-moving inertia-gravity wave solution is not an allowable solution when $n = 0$ because it amplifies away from the Equator (not shown), reducing the number of allowable solutions to (10) by one. The final solution to (10) gives the dispersion relation for mixed Rossby-gravity waves. For convenience, we simply write it as:

(11) $$\omega = \frac{k\sqrt{g'H}}{2}\left(1 - \left(1 + \frac{4\beta}{k^2\sqrt{g'H}}\right)^{1/2}\right)$$

With the phase speed given by:

(12) $$c_p = \frac{\sqrt{g'H}}{2}\left(1 - \left(1 + \frac{4\beta}{k^2\sqrt{g'H}}\right)^{1/2}\right)$$

The $\frac{4\beta}{k^2\sqrt{g'H}}$term in (11) and (12) is positive-definite since each of its components are all positive. Thus, $1 + \frac{4\beta}{k^2\sqrt{g'H}} > 1$, such that its square root is also greater than 1. This means that the phase speed for mixed Rossby-gravity waves is negative, describing *westward-propagating* waves.

For small k (longer wavelengths), the $k\sqrt{g'H}$ term in (11) is relatively small but the $\frac{4\beta}{k^2\sqrt{g'H}}$ term is relatively large. Because k is squared in the latter term, it has a greater influence on ω. In this scenario, ω is relatively large and the wave more closely resembles an inertia-gravity wave. For large k (shorter wavelengths), the inverse is true: ω is relatively small and the wave more closely resembles an equatorial Rossby wave.

Mixed Rossby-gravity waves are characterized by alternating clockwise and counterclockwise rotation that is centered on the Equator. Clockwise rotation is associated with high pressure in the Northern Hemisphere and low pressure in the Southern Hemisphere; the opposite is true for counterclockwise rotation. Horizontal wind is nearly meridional and has its peak magnitude along the Equator; it decays rapidly away from there. In a moist environment, deep, moist convection predominantly forms in regions of speed convergence, or to the east of low pressure. As with equatorial Rossby waves, deep, moist convection redistributes positive potential vorticity from higher to lower altitudes, increasing cyclonic flow to the east of low pressure. This tugs at the wave, slowing its westward propagation. Unlike with equatorial Rossby waves, this also results in mixed Rossby-gravity waves being tilted horizontally toward low pressure.

Kelvin Waves

As Kelvin waves have no meridional motion, (2) – which is derived from the equation describing meridional motions – does not hold for them, necessitating another method for deriving their dispersion relation. This derivation is provided in Appendix B, with a result of:

(13) $$\omega = \pm k\sqrt{g'H}$$

This is identical to the dispersion relation of a pure gravity wave and can be obtained for small n and large k in (7), such that the equatorial Kelvin wave can be viewed as a special case of a gravity wave.

As the negative root for ω in (13) does not decay away from the Equator (and, in fact, leads to amplification of the wave away from the Equator; not shown), we discard it as an allowable solution. Thus, the dispersion relation for Kelvin waves is given only by the positive root of (13).

The phase speed for Kelvin waves is given by:

(14) $$c_p = \sqrt{g'H}$$

Because $g'H$ is positive-definite, $c_p > 0$, such that Kelvin waves propagate *eastward*.

Kelvin waves are characterized by zonal velocity that is strongest at the Equator and decays poleward away from there and where the wave's pressure anomalies have their largest magnitude and decays east and west from there. Pressure anomalies can be understood using shear vorticity: a stick in the wind will rotate in a cyclonic fashion for westerly flow and in an anticyclonic fashion for easterly flow. Deep, moist convection preferentially forms along the Equator where convergence is maximized west of low pressure and east of high pressure. As with the other equatorial waves, the associated vertical redistribution of positive potential vorticity from higher to lower altitudes tugs at the wave and slows its eastward progression.

Equatorial Wave Monitoring

Equatorial waves in the real atmosphere almost always occur in conjunction with deep, moist convection, the horizontal extent, temporal frequency, and propagation direction of which vary between different wave types. Statistical filtering techniques, specifically Fourier transforms, allow us to leverage this information to identify equatorial waves from satellite data.

Fourier transforms allow us to write any continuous function that varies in time and/or space in terms of an infinite integral of waves with varying frequencies (time-varying data) and/or wavenumbers (space-varying data). For instance, let's assume that there is some globally varying weather field at the Equator that can be represented by a bell or Gaussian curve. The Fourier transform of these zonally varying data is an infinite sum of sine and cosine waves, each representing different horizontal scales with different amplitudes. The waves' amplitudes provide a measure of the signal contained at each horizontal scale. If the data varied in time instead of space, the Fourier transform remains an infinite sum of sine and cosine waves, except each representing different temporal scales (or periods) with different amplitudes. The waves' amplitudes in this case provide a measure of the signal contained at each temporal scale.

Consider the bell curve example contained within the lecture materials. The initial curve is relatively wide, with an amplitude of 100 centered at x=50. Applying a Fourier transform to these data returns data that vary in terms of their wavenumber, where wavenumber 0 represents a flat line, wavenumber 1 represents a curve with one wavelength across the domain, wavenumber 2 represents a curve with two wavelengths across the domain, and so on. Filtering these data to depict only one wavenumber at a time, we see that wavenumber 0 is a flat line with an amplitude of about 20, wavenumber 1 has one wavelength with an amplitude of ±40, wavenumber 2 has two wavelengths with an amplitude of ±25, and so on. Filtering these data again so that they depict partial sums of waves (wavenumbers 0+1, wavenumbers 0+1+2, etc.), retaining more waves in the partial sum leads to a curve that looks more like the original bell curve. The partial sum of wavenumbers 0-8 for this example is sufficient to nearly match the original bell curve; note, however, that this will differ from one dataset to another depending on the scales (time and/or space) of the data represented by the data.

In the context of equatorial-wave monitoring, Fourier transforms in both space (specifically, the x-direction at different latitudes near the Equator) and time applied to three-dimensional (x, y, t) satellite data allow us to identify the horizontal scales (wavenumbers), temporal frequencies (periods), and propagation directions (eastward for positive wavenumbers, westward for negative wavenumbers) containing the strongest signals. We can do this separately for features that are symmetric and asymmetric across the Equator. In the tropics, we find that the strongest signals correspond to equatorial waves and other phenomena (the Madden-Julian Oscillation and tropical disturbances) we will consider later in the course (Wheeler and Kiladis 1999).

With this in mind, we can apply a Fourier transform to real-time, three-dimensional satellite data, filter the data to retain only the wavenumbers and frequencies associated with a given wave type, and use a backward Fourier transform to convert the filtered data back into three-dimensional (x, y, t) space. The resulting data represent the signal uniquely associated with a given wave type. When we plot these data, we often contour or shade them only where the returned values exceed a specified threshold. Although there is no universally accepted set of threshold values for this, keep in mind that equatorial waves are inherently associated with relatively weak perturbations. The chosen contour intervals reflect this characteristic in the field (commonly outgoing longwave radiation anomalies, wherein deep, moist convection traps outgoing longwave radiation in the atmosphere and thus is associated with negative outgoing longwave radiation anomalies) that we use to identify equatorial waves.

For Further Reading

- Chapter 4, *An Introduction to Tropical Meteorology, 2nd Edition*, A. Laing and J.-L. Evans, 2016.
- Chapter 7, An Introduction to Dynamic Meteorology, 3rd Edition, J. R. Holton, 1992.

- Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, **44**, 25–43.
- Wheeler, M., and G. N. Kiladis, 1999: Convectively coupled equatorial waves: analysis of clouds and temperature in the wavenumber-frequency domain. *J. Atmos. Sci.*, **56**, 374–399.