# Synoptic Meteorology I: Hydrostatic Balance, the Hypsometric Equation, and Thickness

## For Further Reading

Section 1.4 of *Midlatitude Synoptic Meteorology* by G. Lackmann derives the hypsometric equation and introduces thickness and its applications. Section 3.1 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides a basic derivation of the hydrostatic equation and a full derivation of the hypsometric equation. Section 6-1 of *Weather Analysis* by D. Djurić provides an in-depth discussion of how thickness may be used to identify fronts. Most other dynamic meteorology texts also include derivations and discussions of the hydrostatic and hypsometric equations.

## Derivation of the Hydrostatic Equation

Consider a unit volume ($V$ = 1 $m^3$, with sides of 1 m each) of air in the troposphere that is ***at rest***. Assume that the horizontal properties of the air within the volume are uniform. In the absence of friction and the vertical component of the Coriolis force, there are two forces acting in the vertical on this unit volume of air: one related to pressure and one related to the weight of the air volume (or, more specifically, to gravity).

Recall that:

$$F = pA \tag{1}$$

$$W = gm = g\rho V \tag{2}$$

In (1) and (2), $F$ = force (N), $p$ = pressure (Pa), $A$ = area ($m^2$), $W$ = weight (kg m $s^{-2}$ = N), $g$ = 9.81 m $s^{-2}$ (the gravitational constant), $V$ = volume ($m^3$), and $\rho$ = density (kg $m^{-3}$). The unit N refers to a Newton, equivalent to 1 kg m $s^{-2}$. Consequently, weight $W$ is equivalent to a force.

Consider the forces acting on our air volume (Fig. 1). There are two forces acting in the *downward* direction: weight, or the force associated with gravity ($g\rho A\Delta z$, noting that $V = A\Delta z$ by definition), and the pressure force acting upon the top of the air volume ($p(z+\Delta z)A$). There is a single force acting in the *upward* direction: the pressure force acting upon the bottom of the air volume ($p(z)A$). For both pressure forces, the quantities in parentheses denote the altitudes at which the pressure is valid, not multiplication operators.

In the atmosphere, for an atmosphere at rest (but also frequently also when the atmosphere is not at rest) the upward and downward forces are said to ***balance*** or cancel each other out. We can write this mathematically as:

$$p(z)A - p(z+\Delta z)A - g\rho A\Delta z = 0 \tag{3}$$

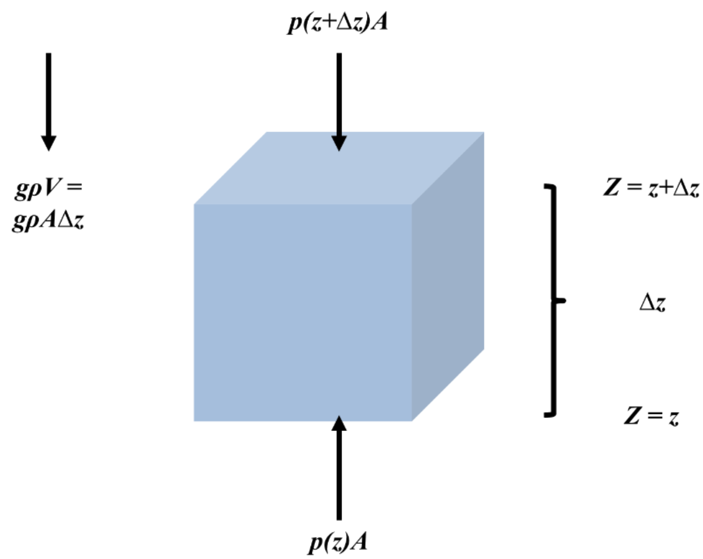


**Figure 1**. Graphical depiction of the forces acting in the vertical direction upon a unit volume of air. In the above, *p(z)* refers to the pressure at an altitude $Z = z$ while *p(z+Δz)* refers to the pressure at an altitude $Z = z + \Delta z$. In this example, volume $V$ is written as the product of the area $A$ and the height of the air volume $\Delta z$.

Note that the upward-directed force is prefaced with a positive sign and that the downward-directed forces are prefaced with negative signs. The idea of balance means that the addition of these forces must be equal to zero, such that the right-hand side of (3) is simply 0.

Next, divide (3) by $A\Delta z$ and group the pressure force terms to obtain:

$$\frac{p(z) - p(z + \Delta z)}{\Delta z} = \rho g \tag{4}$$

Multiplying by -1, we obtain:

$$\frac{p(z + \Delta z) - p(z)}{\Delta z} = -\rho g \tag{5}$$

If we take the limit of (5) as $\Delta z$ approaches 0, we obtain:

$$\lim_{\Delta z \to 0} \frac{p(z + \Delta z) - p(z)}{\Delta z} = -\rho g \tag{6}$$

The left-hand side of (6) is equivalent to the partial derivative of $p$ with respect to $z$, such that:

$$\frac{\partial p}{\partial z} = -\rho g \quad (7)$$

Equation (7) is the ***hydrostatic equation***. It provides a formulaic representation of what is known as ***hydrostatic balance***, describing the balance between the downward-directed gravitational force and the upward-directed pressure gradient force. Recall that the pressure gradient force is always directed from higher pressure toward lower pressure. Since pressure is a function of the mass of air that is above you, pressure is highest at ground level and decreases upward from there. Thus, the vertical component of the pressure gradient force is *always* directed upward.

Newton's Second Law of Motion states that the net force that is imposed on an object is equal to its mass times its acceleration. Stated differently, an object's acceleration is equal to the net force imposed upon the object divided by the object's mass. Under the constraint of hydrostatic balance, where the net force in the vertical direction is zero, air does not accelerate upward or downward. In practice, we find that hydrostatic balance holds when vertical motion is small or weak. This is often true on the synoptic scale but not in thunderstorms, which is beyond the scope of this class.

An alternative derivation of the hydrostatic equation can be obtained by performing a scale analysis of the vertical momentum equation for synoptic-scale motions. This derivation has the advantage of quantitively justifying neglecting friction and the vertical component of the Coriolis force in the derivation; however, it does not start from basic physical principles as does the derivation here.

**Derivation of the Hypsometric Equation**

Recall that the ideal gas law applicable when the air contains a non-zero amount of water vapor can be expressed as:

$$p = \rho R_d T_v \quad (8)$$

In (8), $p$ = pressure (Pa), $\rho$ = density (kg m$^{-3}$), $R_d$ = dry air gas constant (287.04 J kg$^{-1}$ K$^{-1}$), and $T_v$ = virtual temperature (K). The virtual temperature can be approximated by $T_v = T(1 + 0.61w)$, where $w$ = mixing ratio of water vapor (kg kg$^{-1}$). For common values of $w$ of < 0.02 kg kg$^{-1}$ within the lower troposphere, $T_v$ is equal to or slightly larger than $T$.

If we solve (8) for $\rho$ and substitute into the hydrostatic equation (7), we obtain:

$$\frac{\partial p}{\partial z} = -\frac{pg}{R_d T_v} \quad (9)$$

Multiplying both sides of (9) by $\partial z$ and dividing both sides of (9) by $p$, we obtain:

$$\frac{\partial p}{p} = -\frac{g\partial z}{R_d T_v} \tag{10}$$

If we solve (10) for $\partial z$ and make the substitution of $\partial(\ln p)$ for $\partial p/p$, we obtain:

$$-\frac{R_d T_v}{g}\partial(\ln p) = \partial z \tag{11}$$

If we integrate (11) between pressure levels $p_1$ and $p_2$, where $p_1 > p_2$, at which the heights are $z_1$ and $z_2$, where $z_1 < z_2$, we obtain:

$$\int_{p_2}^{p_1} \frac{R_d T_v}{g}\partial(\ln p) = \int_{z_1}^{z_2} \partial z \tag{12}$$

Note that we have changed the order of the integration on the left-hand side ($p_2$ to $p_1$), which permits us to drop the leading negative sign.

On the left-hand side of (12), we have two constants with respect to $p$: $R_d$ and $g$. However, $T_v$ is not constant with respect to $p$ – in fact, it is very much not constant with respect to $p$! To simplify the integration, we approximate $T_v$ by a layer-mean value $\overline{T_v}$ that is constant with respect to $p$. If we make this approximation and then integrate both sides of (12), we obtain:

$$\frac{R_d \overline{T_v}}{g}\left(\ln(p_1) - \ln(p_2)\right) = z_2 - z_1 \tag{13}$$

If we combine the natural logarithms in (13) into a single term, we obtain:

$$\frac{R_d \overline{T_v}}{g}\ln\left(\frac{p_1}{p_2}\right) = z_2 - z_1 \tag{14}$$

Equation (14) is the ***hypsometric equation***. Because $p_1 > p_2$, the natural logarithm on the left-hand side of (14) is positive-definite (i.e., is always positive). The constants $R_d$ and $g$ are also positive-definite. This enables us to simply (14) to the following proportionality:

$$\overline{T_v} \propto z_2 - z_1 \tag{15}$$

This means that the difference in height $z_2 - z_1$ between two pressure surfaces $p_1$ and $p_2$, which we refer to as ***thickness*** ($z_2 - z_1 = \Delta z$), is directly proportional to the mean virtual temperature between the two pressure surfaces $p_1$ and $p_2$. This is a powerful statement, one that has many applications to understanding the Earth's atmosphere as well as synoptic-scale meteorological phenomena! We next consider several applications of this relationship.

## Meteorological Applications of the Hypsometric Equation

*The Height of Tropospheric Isobaric Surfaces*

If we take $z_1 = z_{surface} = 0$ m, such that $p_1 = p_{surface}$, then (15) tells us that the height $z_2$ of some isobaric surface $p_2$ within the troposphere is higher when the layer-mean virtual temperature is higher. Conversely, the height $z_2$ of the isobaric surface $p_2$ is lower when the layer-mean virtual temperature is lower. Consider, for example, the 500 hPa isobaric surface. The height of the 500 hPa isobaric surface is higher where the layer-mean temperature is higher and lower where the layer-mean temperature is lower. When the layer-mean temperature rapidly changes in the zonal and/or meridional directions, so too does the height of the 500 hPa isobaric surface.

Let's apply this concept. Due to the annually averaged incoming solar radiation imbalance between the poles and Equator, air temperature at and above the surface is typically coldest at the poles and increases as you move toward the Equator. Thus, on planetary scales, we expect the height of the 500 hPa isobaric surface to be highest at the Equator and lowest at the poles. This is why the 500-hPa height (or the height of any tropospheric isobaric surface, for that matter) is typically highest in the tropics and lowest at polar latitudes.

We can also apply this concept on the synoptic scale. Two hypothetical rawinsonde observations indicate that the 1000-hPa height is 100 m at both Green Bay, WI and Jacksonville, FL. The 1000-500 hPa $\overline{T_v}$ observed by these rawinsondes is 286 K at Jacksonville and 270 K at Green Bay. From these data, we can use the hypsometric equation to compute the 500 hPa height at each location:

Jacksonville, FL: $\frac{R_d(286\ K)}{g} ln\left(\frac{1000\ hPa}{500\ hPa}\right) = z_2 - 100\ m$, such that $z_2$ = 5900.49 m

Green Bay, WI: $\frac{R_d(270\ K)}{g} ln\left(\frac{1000\ hPa}{500\ hPa}\right) = z_2 - 100\ m$, such that $z_2$ = 5575.99 m

Thus, the 1000-500 hPa thickness is lower at the colder Green Bay than the warmer Jacksonville. A relatively small layer-mean virtual temperature difference of 16 K results in a 324.5 m 500-hPa height difference between the two locations!

*Precipitation Type Analysis and Forecasting*

Rawinsonde observations measure the heights of isobaric surfaces above the ground, data that can be used to analyze the thickness between any two isobaric surfaces, such as 1000 hPa and 500 hPa or 1000 hPa and 850 hPa. Data obtained from numerical weather prediction models can be used to do the same for forecast data. Because the thickness between two isobaric surfaces is directly proportional to the mean virtual temperature in the vertical layer between the two isobaric surfaces, precipitation type may be crudely diagnosed from analyses and forecasts of thickness.

A commonly-used rule of thumb states that when the thickness of the 1000–500 hPa layer is less than 5400 m, snow rather than rain is the most likely precipitation type. We can use (14) to "prove" this rule of thumb. Plugging in 1000 hPa for $p_1$, 500 hPa for $p_2$, 5400 m for $z_2 - z_1$, and the known values for $R_d$ and $g$, we obtain $\overline{T_v}$ = 266.25 K (-6.9°C). Because $T \leq T_v$, the layer-mean temperature between 1000–500 hPa is ≤ -6.9°C. We thus might reasonably expect that it is cold enough in this layer to support snow reaching the surface (assuming that precipitation is possible or occurring). Note, however, that precipitation type is crucially dependent upon the vertical temperature profile between cloud base and the ground, such that a full diagnosis of precipitation type requires analysis of observed or forecast skew *T*-ln *p* diagrams.

*The Vertical Structure of Cyclones and Anticyclones*

Thickness is a powerful tool by which the vertical structure of both cyclones and anticyclones may be understood. Let us consider two examples…

a) "An area of low pressure at the surface found within a warm air column disappears quickly with height."

Earlier, we stated that the thickness of the layer between two isobaric surfaces is directly proportional to the mean virtual temperature within that layer. Here, we have a warm air column, and thus we would expect the thickness within this column to be large compared to locations outside of this column. This means that, within the warm air column, isobaric surfaces at the bottom of the layer will be depressed downward toward the ground compared to locations outside of the warm air column. Isobaric surfaces at the top of the layer within the warm air column will be elevated upward compared to locations outside of the warm air column.

Consequently, at and near the surface, the pressure within the warm air column will be lower than outside of the warm air column. However, as you move upward, the pressure within the warm air column becomes larger than outside of the warm air column (Fig. 2). Real-world examples of areas of low pressure at the surface found within columns of warm air include tropical cyclones and heat lows. The relative warmth found at the core of these features, coupled with their reduced intensity with increasing height, give rise to the term *warm-core cyclones* to describe these features.

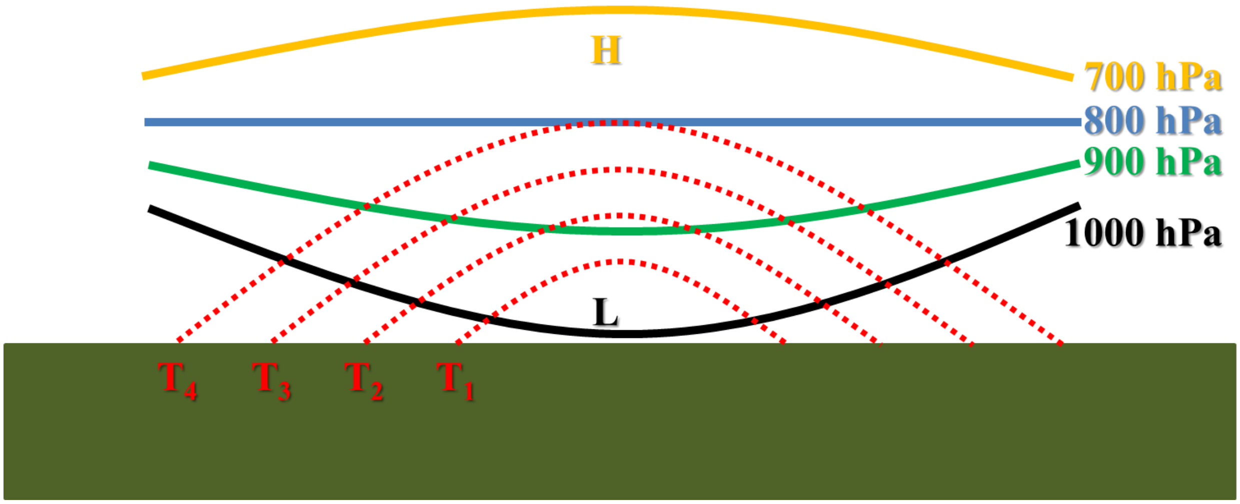


**Figure 2**. Schematic meant to accompany example (a) above. The red lines denote representative isotherms, where $T_1 > T_2 > T_3 > T_4$, such that the warmest temperatures are found in the center of the figure. Four isobaric surfaces, 700 hPa (orange), 800 hPa (blue), 900 hPa (green), and 1000 hPa (black), are given by solid lines. Note the greater vertical spacing between the isobaric surfaces within the warm air column as compared to outside the warm air column. Thus, an area of locally lower pressure at and near the surface weakens and disappears with increasing altitude.

b) "An area of low pressure at the surface found within a cold air column increases in intensity with height."

This is the opposite of what we described in the previous example, with an area of low pressure now located in a cold-air column. We expect the thickness of a layer between two isobaric surfaces within this column to be lower than between the same two isobaric surfaces outside of this column. Consequently, the low pressure becomes stronger with increasing altitude (Fig. 3).

Real-world examples of areas of low pressure at the surface found within cold air columns include mid-latitude cyclones, or those associated with fronts, that we will study extensively this semester and next. The relatively coolness found at the heart of these features, coupled with their increasing intensity with increasing height, give rise to the term *cold-core cyclones* to describe these features.

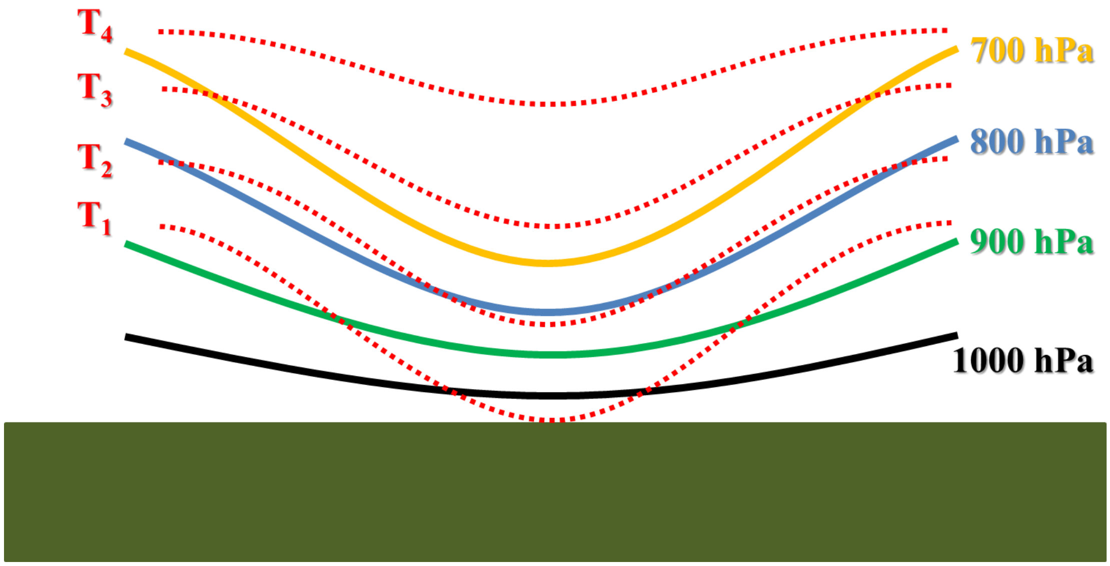


**Figure 3**. Schematic meant to accompany example (b) above. The red lines denote representative isotherms, where $T_1 > T_2 > T_3 > T_4$, such that the coldest temperatures are found in the center of the figure. Four isobaric surfaces, 700 hPa (orange), 800 hPa (blue), 900 hPa (green), and 1000 hPa (black), are given by solid lines. Note the smaller vertical spacing between the isobaric surfaces within the cold air column as compared to outside the cold air column. Thus, an area of locally lower pressure at and near the surface becomes more intense with increasing altitude