Synoptic Meteorology II: The Non-Conservation of Isentropic Potential Vorticity

Readings: Sections 4.3.1 and 4.3.2 of *Midlatitude Synoptic Meteorology*

Mathematical Formulation

In our isentropic potential vorticity derivation, we stated that the vorticity equation applicable on isentropic surfaces (neglecting friction and diabatic heating) takes the form:

$$\frac{\partial \zeta}{\partial t} + \vec{\mathbf{v}} \cdot \nabla_\theta (\zeta + f) + (\zeta + f)\nabla_\theta \cdot \vec{\mathbf{v}} = 0 \tag{1}$$

If we substitute $\eta = \zeta + f$, noting that f is constant with respect to both t and θ, and add in the effects of friction (one term) and diabatic heating (two terms) to the right side of (1), we obtain:

$$\frac{\partial \eta}{\partial t} + \vec{\mathbf{v}} \cdot \nabla_\theta \eta + \eta(\nabla_\theta \cdot \vec{\mathbf{v}}) = -\dot{\theta}\frac{\partial \eta}{\partial \theta} + \hat{\mathbf{k}} \cdot \left(\frac{\partial \vec{\mathbf{v}}}{\partial \theta} \times \nabla_\theta \dot{\theta}\right) + \hat{\mathbf{k}} \cdot \left(\nabla_\theta \times \vec{\mathbf{F}}\right) \tag{2}$$

Also from that derivation, we stated that the continuity equation applicable on isentropic surfaces (neglecting diabatic heating) takes the form:

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial p}\right) - \frac{\partial \theta}{\partial p}(\nabla_\theta \cdot \vec{\mathbf{v}}) + \vec{\mathbf{v}} \cdot \nabla_\theta \left(\frac{\partial \theta}{\partial p}\right) = 0 \tag{3}$$

If we add the two diabatic heating terms to the right side of (3), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial p}\right) - \frac{\partial \theta}{\partial p}(\nabla_\theta \cdot \vec{\mathbf{v}}) + \vec{\mathbf{v}} \cdot \nabla_\theta \left(\frac{\partial \theta}{\partial p}\right) = \frac{\partial \theta}{\partial p}\frac{\partial \dot{\theta}}{\partial t} - \dot{\theta}\frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial p}\right) \tag{4}$$

We now wish to obtain an expression for the total derivative of P. Multiplying (2) by $-g\partial\theta/\partial p$ and adding to it (4) multiplied by $-g\eta$ allows us to eliminate the divergence term $(\nabla_\theta \cdot \vec{\mathbf{v}})$ found in both equations. If we do this, use the chain rule to combine terms, and simplify, we obtain:

$$\frac{DP}{Dt} = -g\eta\frac{\partial \dot{\theta}}{\partial p} - g\frac{\partial \theta}{\partial p}\hat{\mathbf{k}} \cdot \left(\frac{\partial \vec{\mathbf{v}}}{\partial \theta} \times \nabla_\theta \dot{\theta}\right) - g\frac{\partial \theta}{\partial p}\hat{\mathbf{k}} \cdot \left(\nabla_\theta \times \vec{\mathbf{F}}\right) \tag{5}$$

(5) describes the *non-conservation* of isentropic potential vorticity (i.e., changes in P following the flow) as a function of diabatic heating – or, more specifically, vertical and horizontal gradients thereof – and friction.

The first forcing term on the right side of (5) is known as the *vertical diabatic* term because it is related to the vertical structure of diabatic heating. The second forcing term on the right side of

(5) is known as the *shear diabatic* term because it is related to the vertical wind shear (change in wind speed and direction across isentropic levels) and the horizontal gradient of diabatic heating. The last forcing term on the right side of (5) is simply the *friction* term.

Interpretation of Diabatic Heating and Frictional Impacts upon IPV

Vertical Diabatic Term

In isolation, the contribution of the vertical diabatic term to changes in IPV can be expressed as:

$$\frac{DP}{Dt} \approx -g\eta \frac{\partial \dot{\theta}}{\partial p} \tag{6}$$

From (6), it is apparent that we need to know the signs of η and $\partial\dot{\theta}/\partial p$ to diagnose changes in isentropic potential vorticity following the motion.

Consider a midtropospheric diabatic-warming ($\dot{\theta} > 0$) maximum, as depicted in Fig. 1. A representative example of such a diabatic-warming distribution is given by latent heat release due to condensation. Such a maximum is often located in the midtroposphere, where ascent can be strong while water-vapor content is not overly small.

Below the level of the $\dot{\theta}$ maximum, $\dot{\theta}$ increases upward. Conversely, above the level of the $\dot{\theta}$ maximum, $\dot{\theta}$ decreases upward. Since pressure decreases upward, $\partial\dot{\theta}/\partial p < 0$ below the level of the $\dot{\theta}$ maximum and $\partial\dot{\theta}/\partial p > 0$ above the level of the $\dot{\theta}$ maximum. If this occurs in concert with cyclonic absolute vorticity ($\eta > 0$), then since $g > 0$, P will increase following the motion below and decrease following the motion above the level of the $\dot{\theta}$ maximum!

We can apply this to surface cyclogenesis. The thought experiment above demonstrates that midtropospheric diabatic warming occurring above a surface cyclone will increase P following its motion. Therefore, the effect of diabatic warming upon surface cyclogenesis is the same as that in the Petterssen-Sutcliffe development framework: it acts to enhance surface cyclogenesis!

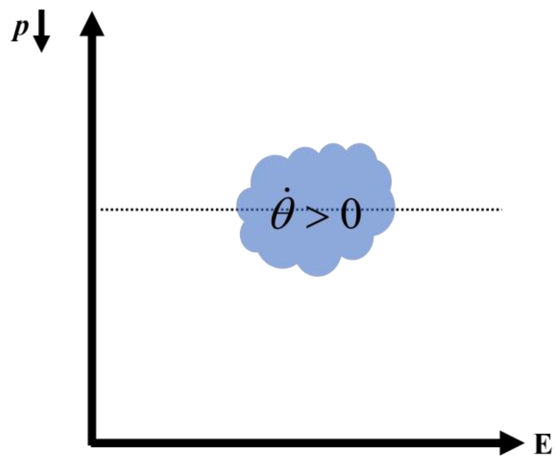

Figure 1. East-west vertical cross-section of isentropes (red lines) in the presence of a middle tropospheric maximum in diabatic warming (blue cloud).

We can confirm these insights by considering how vertically localized diabatic warming influences IPV's components. We start with static stability. Since θ generally increases with height, diabatic warming forces isentropes downward. This is depicted graphically in Fig. 2. Here, the greatest warming is concentrated along the θ_3 isentrope, which is nudged downward through the warming maximum. This increases isentropes' vertical packing below the warming maximum and decreases it above, consistent with our earlier inference of increased P below and decreased P above the warming maximum.

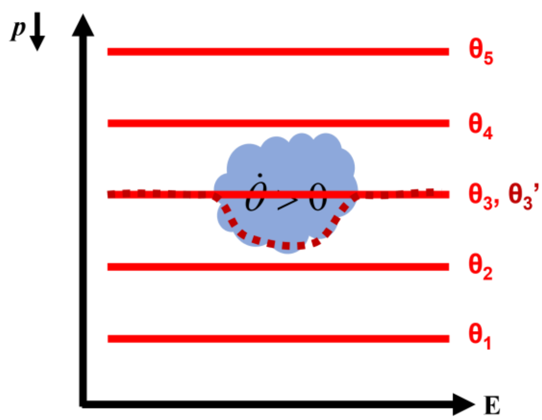

Figure 2. As in Fig. 1, except depicting isentropes at a later time post-warming.

The impacts of vertically localized diabatic warming on absolute vorticity can be examined using two complementary perspectives…

1) *Quasi-geostrophic perspective*: From the omega equation, $\dot{\theta}$ is proportional to $-\omega$. Thus, diabatic warming forces ascent ($\omega < 0$) that is maximized at the level of the diabatic-warming maximum. If we assume that $\omega \approx 0$ at the surface and tropopause, since $\partial p < 0$, $\partial\omega/\partial p$ is positive below and negative above the level of the diabatic-warming maximum. The quasi-geostrophic vorticity equation indicates that the change of geostrophic relative vorticity – a major contributor to the absolute vorticity – is proportional to $\partial\omega/\partial p$. Thus, η (and, by extension, P) increases below and decreases above the level of the diabatic-warming maximum.

2) *Thickness perspective*: Diabatic warming increases the potential temperature of a vertical layer, increasing its thickness. This forces pressure surfaces downward below and upward above the level of maximum diabatic warming, creating locally low heights in the lower troposphere and locally higher heights in the upper troposphere. Given the relationship between geopotential height and geostrophic relative vorticity (and thus absolute vorticity), this is associated with increased η below and decreased η above the level of the diabatic-warming maximum.

Now, consider a slightly more complex example: condensation (and diabatic warming) occurring above a layer of evaporation (and diabatic cooling). Such a configuration is often found in the stratiform rain region of mesoscale convective systems and in the isentropic upglide region along and to the north of a warm front. This vertical profile of diabatic heating results in decreased IPV above the diabatic-warming maximum and below the diabatic-cooling maximum (both locations where $\dot{\theta}$ decreases upward) and increased IPV between the diabatic-warming and diabatic-cooling maxima (both locations where $\dot{\theta}$ increases upward). This diabatic-heating configuration is known to result in the formation of midtropospheric mesoscale convective vortices in an MCS's trailing stratiform region.

Shear Diabatic Term

In isolation, the contribution of the shear diabatic term to changes in IPV can be expressed as:

$$\frac{DP}{Dt} \approx -g\frac{\partial\theta}{\partial p}\hat{\mathbf{k}}\cdot\left(\frac{\partial\vec{\mathbf{v}}}{\partial\theta}\times\nabla_{\theta}\dot{\theta}\right) \tag{7}$$

If we expand (7) into its components, we obtain:

$$\frac{DP}{Dt} \approx -g\frac{\partial\theta}{\partial p}\left(\frac{\partial u}{\partial\theta}\frac{\partial\dot{\theta}}{\partial y} - \frac{\partial v}{\partial\theta}\frac{\partial\dot{\theta}}{\partial x}\right) \tag{8}$$

To interpret this, consider the case where there is southwesterly vertical wind shear. In addition, let us presume that there exists a midtropospheric diabatic-warming maximum to the northwest of a surface cyclone. These are depicted in Fig. 3 below.

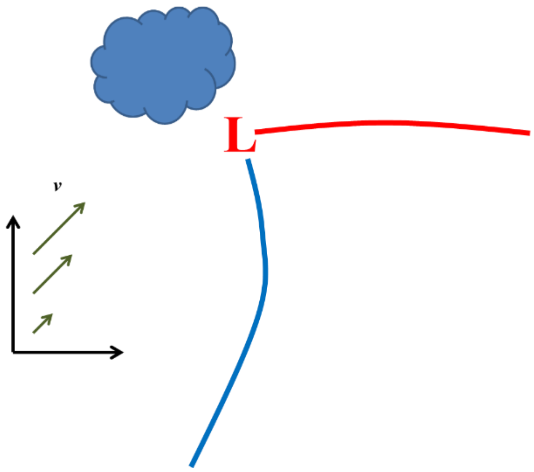

Figure 3. Depiction of a surface cyclone and its warm (red line) and cold (blue line) fronts located to the southeast of a middle tropospheric diabatic warming maximum (blue cloud). The vertical wind shear is denoted by the inset at left.

For southwesterly shear, as both u and v become more positive (westerly, southerly) upward, both $\partial u/\partial\theta$ and $\partial v/\partial\theta$ are positive (assuming θ also increases upward). Since the vertical wind shear is purely southwesterly (i.e., $u = v$ at all heights), $\partial u/\partial\theta = \partial v/\partial\theta$ in this example. Assume that the atmosphere is statically stable, such that $-\partial\theta/\partial p > 0$. Thus, given the sign convention on $\partial u/\partial\theta$ and $\partial v/\partial\theta$ for this case, $DP/Dt > 0$ for $\partial\dot{\theta}/\partial y > 0$ and $\partial\dot{\theta}/\partial x < 0$. Conversely, $DP/Dt < 0$ for $\partial\dot{\theta}/\partial y < 0$ and $\partial\dot{\theta}/\partial x > 0$.

Since the positive x-axis is directed toward the east, $\partial\dot{\theta}/\partial x$ is negative east and positive west of the diabatic warming maximum. Likewise, since the positive y-axis is directed toward the north, $\partial\dot{\theta}/\partial y$ is negative north and positive south of the diabatic warming maximum. For simplicity, we assume that the magnitudes of $\partial\dot{\theta}/\partial x$ and $\partial\dot{\theta}/\partial y$ are equal everywhere. Thus, ***for the case of southwesterly vertical wind shear,*** $DP/Dt < 0$ in the midtropospheric northwest and $DP/Dt > 0$ in the midtropospheric southeast of the diabatic-warming maximum.

More generally, for a localized diabatic-warming maximum with positive static stability, P will decrease following the motion on the left side of the vertical wind shear vector and P will increase following the motion on the right side of the vertical wind shear vector.

Applications of Diabatic Heating and its Influences on IPV

By definition, diabatically altering the IPV must result in changes to its underlying kinematic and thermodynamic fields. This, in turn, can influence quantities such as the direction and magnitude of the vertical wind shear and, through the thermal wind relationship, the magnitude of the horizontal temperature gradient over that layer.

This is particularly important for tropical transition, or the transformation of a midlatitude, non-tropical, synoptic-scale cyclone into a tropical cyclone. In the context of the vertical diabatic term, a midtropospheric diabatic-warming maximum decreases P following the motion in the upper troposphere and increases P following the motion in the lower troposphere. Decreasing P aloft weakens the magnitude of the horizontal P gradient – and thus the wind speed – atop the diabatic-warming maximum. Increasing P near the surface increases the magnitude of the horizontal P gradient – and thus the wind speed – below the diabatic-warming maximum. Together, these counteract the typical increase in wind speed with height in midlatitude environments (i.e., reduced vertical wind shear), in turn reducing the magnitude of the horizontal layer-mean temperature gradient (through the thermal wind relationship). These changes are necessary (although insufficient) conditions for a tropical cyclone to form.

IPV non-conservation often plays an important role in the development of both midlatitude and tropical cyclones, as we discuss in more detail in this and later lectures. However, IPV non-conservation is not always well-forecasted or represented by numerical weather prediction models. Why? Diabatic heating is difficult to represent accurately in a forecast model: it involves numerous complex physical processes that we cannot faithfully represent in the model, nor can we observe well with existing instruments. Thus, errors in representing diabatic heating that are inevitably present within model forecasts can and often do lead to errors in the rest of the forecast – and, for some systems, such errors can be quite large and meaningful!

Frictional Term

In isolation, the contribution of the frictional term to changes in IPV can be expressed as:

$$\frac{DP}{Dt} = -g\frac{\partial \theta}{\partial p}\hat{\mathbf{k}} \cdot \left(\nabla_{\theta} \times \vec{\mathbf{F}}\right) \tag{9}$$

As in the quasi-geostrophic system, friction acts as a *brake* on the intensity of boundary-layer IPV anomalies, whether cyclonic or anticyclonic in nature.

A Cautionary Note on the Physical Interpretation of IPV Non-Conservation

By use of the divergence (or Gauss') theorem, it can be shown that:

$$\int_V \frac{\partial(\sigma P)}{\partial t} dV = 0 \tag{10}$$

The σ in (10) is again a measure of static stability. This equation states that the volume-integrated change of P with respect to time is zero.

Presuming that an IPV anomaly – or, more accurately, the effects of diabatic heating and friction on that anomaly – are isolated from the external environment, the *volume-integrated* P **does not change** due to diabatic heating or friction. Instead, P is rearranged both horizontally and vertically within the volume to result in local changes in P!

Let us view this in the context of our midtropospheric diabatic-warming maximum in Fig. 1. In this example, diabatic warming increases P in the lower troposphere and decreases P in the upper troposphere. Rather than creating or destroying IPV, however, the diabatic heating ***rearranges*** P. Since P typically increases upward, this can be viewed as bringing higher P downward and lower P upward. Similar arguments can be made for diabatic cooling (for this term) or for the shear diabatic term (for any diabatic heating). Therefore, it is not appropriate to view the effects of diabatic heating (and friction) upon the non-conservation of P in the context of IPV creation or destruction – rather, one should view them in the context of the horizontal and vertical rearrangement of P.

Evolution of Upper-Tropospheric IPV Anomalies

Decay of an Upper-Tropospheric Positive IPV Anomaly

Consider the decay of an upper-tropospheric positive IPV anomaly. This anomaly is strongest in the upper troposphere but is still present (in a weakened form) downward toward the surface.

When we related surface potential-temperature anomalies to upper-tropospheric IPV anomalies, we introduced thermal vorticity. Thermal vorticity relates the change of the geostrophic relative vorticity with height to the Laplacian of the potential temperature field, as expressed by:

$$f_0 \frac{\partial \zeta_g}{\partial p} = -h\nabla^2\theta \qquad (11)$$

Because the positive IPV anomaly and its associated cyclonic rotation increase in intensity ($\partial\zeta_g > 0$) with height ($\partial p < 0$), the left side of (11) is negative in the Northern Hemisphere ($f_0 > 0$). Due to the leading negative on the right side of (11), and with h being positive, $\nabla^2\theta$ thus is positive, such that θ is a local minimum.

We know, however, that a localized θ minimum at the surface is akin to an upper-tropospheric negative (or anticyclonic) IPV anomaly. Similar to its upper-tropospheric counterpart, this surface cold potential-temperature anomaly is strongest at the surface but is still present (in a weakened form) upward from the surface. Note, however, the opposing signs of the lower- and upper-tropospheric anomalies! The two counteract each other, such that over a sufficiently long period of time, the upper-tropospheric positive IPV anomaly becomes self-destructive. This process can be hastened in the presence of midtropospheric diabatic warming, as illustrated earlier in this lecture.

Decay of an Upper Tropospheric Negative IPV Anomaly

Consider the decay of an upper-tropospheric negative IPV anomaly. This anomaly is strongest in the upper troposphere but is still present (in a weakened form) downward toward the surface.

Because the negative IPV anomaly and its associated anticyclonic rotation increase in intensity ($\partial\zeta_g < 0$) with height ($\partial p < 0$), the left side of (11) is positive in the Northern Hemisphere ($f_0 > 0$). Due to the leading negative on the right side of (11), and with h being positive, $\nabla^2\theta$ thus is negative, such that θ is a local maximum. The resulting surface θ maximum is akin to an upper-tropospheric positive (or cyclonic) IPV anomaly, counteracting the negative upper-tropospheric IPV anomaly. Thus, over a sufficiently long period of time, the upper-tropospheric negative IPV anomaly becomes self-destructive. This process can be hastened in the presence of midtropospheric diabatic cooling.