Synoptic Meteorology II: The Quasi-Geostrophic Vorticity Equation

Readings: Section 2.2 of *Midlatitude Synoptic Meteorology*.

Refresher on the Geostrophic Approximation

Last semester, we introduced the Rossby number:

$$Ro = \frac{acceleration}{Coriolis} = \frac{\frac{U^2}{L}}{f_0 U} = \frac{U}{f_0 L} \tag{1}$$

We also introduced characteristic values of *U*, *L*, and f_0 for midlatitude, synoptic-scale motions:

Variable	Characteristic Value	Description
u, v	U ≈ 10 m s^{-1}	Horizontal velocity scale
x, y	L ≈ 10^6 m	Length scale
f	$f_0 \approx 10^{-4}$ s^{-1}	Coriolis scale

Based on these values, we find that the characteristic Rossby number for midlatitude, synoptic-scale motions is 0.1. This defines geostrophic balance, representing the force balance between the horizontal pressure gradient and Coriolis forces:

$$\vec{\boldsymbol{v}}_g = \frac{1}{f}\hat{\boldsymbol{k}} \times \nabla_p \Phi \tag{2}$$

where the geopotential $\Phi = gz$, with horizontal gradients of *z* on an isobaric surface corresponding to horizontal gradients of *p* on a constant height surface. At the same time, (2) denotes that along-flow accelerations have small magnitudes as compared to the magnitudes of the horizontal pressure gradient and Coriolis forces and thus can be neglected.

The total wind can be decomposed into its geostrophic and ageostrophic components:

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_g + \vec{\boldsymbol{v}}_{ag} \tag{3}$$

Because along-flow accelerations are neglected in the geostrophic wind, along-flow accelerations are uniquely associated with the ageostrophic wind. Returning to (1), this allows us to state that the ageostrophic wind $\mathbf{v}_{ag}$ is much smaller than the geostrophic wind $\mathbf{v}_g$, such that:

$$\frac{\|\vec{\boldsymbol{v}}_{ag}\|}{\|\vec{\boldsymbol{v}}_g\|} = O(Ro) \tag{4}$$

The Quasi-Geostrophic Approximation

There are three principles underlying the quasi-geostrophic approximation:

1) Advection is primarily horizontal and dominated by the geostrophic wind.

   This allows us to write the total derivative as a function of only the geostrophic wind, i.e.,

$$\frac{D}{Dt} \approx \frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

   Since the geostrophic wind is entirely horizontal, the total derivative does not contain any vertical motion terms (which are exclusively associated with ageostrophic motions).

2) The Coriolis parameter at a given latitude ϕ_0 can be expressed in terms of a constant value f_0 plus a small perturbation that is based on the product of the local meridional variability in f and the physical distance from ϕ_0, i.e.,

$$f = f_0 + \beta y, \text{ where } \beta = \left.\frac{\partial f}{\partial y}\right|_{\phi_0} = \frac{2\Omega \cos\phi_0}{a}$$

   The βy term is approximately one order of magnitude less than f_0, which allows us to write (2) in terms of f_0 rather than f. This is equivalent to stating that the meridional length scale of the feature being studied is small relative to the Earth's radius. In addition, substituting f with f_0 makes the geostrophic wind in (2) entirely non-divergent. Although we previously assumed this to be the case, this is only formally true when f is constant.

3) Localized (in time and space) vertical temperature perturbations are negligibly small.

Altogether, these approximations help to simplify the complex dynamics of synoptic-scale weather systems without a significant loss of accuracy.

The Quasi-Geostrophic Primitive Equations

To apply principles of quasi-geostrophic theory to the study of midlatitude, synoptic-scale weather systems, we must rewrite the full primitive equations.

Horizontal Momentum Equations

For purely geostrophic flow, the horizontal momentum equations simplify to geostrophic balance. However, the flow is never exclusively geostrophic, such that the horizontal momentum equations are somewhat more complex.

Assume that advection by the ageostrophic wind and that changes in the ageostrophic wind along the flow are small. Neglecting curvature terms and substituting $\vec{\boldsymbol{v}} = \overrightarrow{\boldsymbol{v}_g} + \overrightarrow{\boldsymbol{v}_{ag}}$ into the horizontal momentum equations, we can obtain the quasi-geostrophic form of the horizontal momentum equations. This is given by:

$$\frac{D_g\vec{\boldsymbol{v}}_g}{Dt} = -f_0\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{ag} - \beta y\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_g + \vec{\boldsymbol{F}} \qquad (5)$$

where $f = f_0 + \beta y$, an approximation known as the midlatitude beta-plane approximation. Here, $f_0 = f$ at some reference latitude ϕ_0, $\beta = (df/dy)$ evaluated at ϕ_0, and $y = 0$ at ϕ_0. **F** represents friction and is often neglected, a legitimate assumption above the boundary layer.

The left-hand side of (5) indicates that it describes *changes in the geostrophic wind following the geostrophic flow*. Thus, the total derivative on the left-hand side of (5) takes the general form:

$$\frac{D_g}{Dt}(\quad) \equiv \frac{\partial}{\partial t}(\quad) + \vec{\boldsymbol{v}}_g \cdot \nabla(\quad) \equiv \frac{\partial}{\partial t}(\quad) + u_g\frac{\partial}{\partial x}(\quad) + v_g\frac{\partial}{\partial y}(\quad)$$

Recall that the geostrophic wind is purely horizontal; since the geostrophic wind is non-divergent, vertical motion is uniquely associated with the ageostrophic wind. Thus, the advection term is two-dimensional in the geostrophic total derivative's definition.

The right-hand side of (5) describes three forcings that can change the geostrophic wind following the geostrophic flow: two related to the Coriolis force and one related to friction. If we expand the two Coriolis-related terms into their components, we obtain:

$$-f_0\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{ag} = f_0 v_{ag}\hat{\boldsymbol{\imath}} - f_0 u_{ag}\hat{\boldsymbol{\jmath}}$$
$$-\beta y\boldsymbol{k} \times \vec{\boldsymbol{v}}_g = \beta y v_g\hat{\boldsymbol{\imath}} - \beta y u_g\hat{\boldsymbol{\jmath}}$$

The first term indicates that the ageostrophic wind can change the geostrophic wind following the motion. The second term indicates that the meridional variation in the Coriolis force can change the geostrophic wind following the motion.

Note that (5) does not contain a horizontal pressure gradient term or a standard Coriolis term (i.e., the product of the Coriolis parameter and the geostrophic wind). These forcings *define* geostrophic balance, whereas (5) represents forcings that *change* geostrophic balance.

Because of the relative magnitudes of $\boldsymbol{v}_{ag}$ and βy as compared to $\boldsymbol{v}_g$ and f_0, respectively, the Coriolis terms in (5) are one order of magnitude smaller than the product of f_0 and $\boldsymbol{v}_g$ (the Coriolis force). Since the Coriolis and horizontal pressure gradient forces inherently have equal magnitudes under geostrophic balance, the forcings in (5) are also one order of magnitude smaller than the magnitude of the horizontal pressure gradient force.

Vertical Momentum Equation

The quasi-geostrophic vertical momentum equation is given by the hydrostatic equation:

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} \tag{6}$$

Note that this form of the hydrostatic equation differs slightly from that which we are most familiar with – here, the ideal gas law has been substituted for density, and the definition of the geopotential has been substituted for *gz*.

To interpret (6) in the context of the quasi-geostrophic approximation, it is useful to recall precisely what is meant by the concept of hydrostatic balance. Nominally, hydrostatic balance is defined *in the absence of vertical accelerations* as the balance between the gravitational force and the vertical component of the pressure gradient force. Thus, by invoking hydrostatic balance, we are implicitly stating that synoptic-scale vertical motions are of small magnitude.

Continuity Equation

The quasi-geostrophic form of the continuity equation is given by:

$$\nabla \cdot \vec{\boldsymbol{v}}_{ag} + \frac{\partial \omega}{\partial p} = 0 \tag{7}$$

Why does only the ageostrophic wind appear in the first term of (7)? We previously demonstrated the geostrophic wind to be non-divergent (i.e., $\nabla \cdot \vec{\boldsymbol{v}}_g = 0$). Thus, upon substituting (3) for the full wind in the continuity equation, only the ageostrophic component remains:

$$\nabla \cdot \vec{\boldsymbol{v}} = \nabla \cdot \left(\vec{\boldsymbol{v}}_g + \vec{\boldsymbol{v}}_{ag}\right) = \nabla \cdot \vec{\boldsymbol{v}}_g + \nabla \cdot \vec{\boldsymbol{v}}_{ag} = \nabla \cdot \vec{\boldsymbol{v}}_{ag}$$

The primary deduction from (7) is straightforward: synoptic-scale vertical motion is a function of only the ageostrophic wind. We will return to this principle at times throughout the course.

Thermodynamic Equation

Finally, the quasi-geostrophic form of the thermodynamic equation is given by:

$$\frac{\partial T}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla T - S_p \omega = \frac{1}{c_p} \frac{dQ}{dt} \tag{8}$$

where S_p is the static stability (which can be approximated by $-\partial \theta / \partial p$) and dQ/dt is the diabatic heating rate (e.g., radiative input or loss, moisture phase changes, etc.).

From left to right in (8), the terms represent the local temperature tendency, horizontal advection of temperature by the geostrophic wind, adiabatic warming and cooling, and diabatic warming and cooling. The primary difference between the full and quasi-geostrophic thermodynamic equations is the presence of the geostrophic wind in the horizontal advection term of the latter.

The adiabatic warming/cooling term, $S_p\omega$, reflects temperature changes associated with vertical motions (which, again, are exclusively associated with the ageostrophic wind) under adiabatic (i.e., θ-conserving) conditions. We presume a statically stable atmosphere, or one in which the potential temperature increases upward, such that S_p is positive-definite. Dry-adiabatic ascent ($\omega < 0$) results an air parcel cooling ($\partial T/\partial t < 0$) by adiabatic expansion. Dry-adiabatic descent ($\omega > 0$) results in an air parcel warming ($\partial T/\partial t > 0$) by adiabatic compression. Though vertical motion is typically small on the synoptic-scale, consistent with the ageostrophic wind being an order of magnitude weaker than the geostrophic wind, this term cannot be neglected as the static stability is typically relatively large.

Defining the Geostrophic Relative Vorticity

Recall that the geostrophic wind on isobaric surfaces takes the form given by (2). If we expand (2) into its components, we obtain:

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \tag{9a}$$

$$v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} \tag{9b}$$

Recall that the relative vorticity, or ζ, is given by:

$$\zeta = \hat{\boldsymbol{k}} \cdot (\nabla \times \vec{\boldsymbol{v}}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{10}$$

The geostrophic form of the relative vorticity can be expressed similarly as:

$$\zeta_g = \hat{\boldsymbol{k}} \cdot \left(\nabla \times \overrightarrow{\boldsymbol{v_g}}\right) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \tag{11}$$

Because we have definitions for both u_g and v_g, as given by (9), we can go a step further than this. Plugging (9) into (11), we obtain:

$$\zeta_g = \frac{\partial}{\partial x}\left(\frac{1}{f_0}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f_0}\frac{\partial \Phi}{\partial y}\right) \tag{12}$$

Because f_0 is a constant with respect to both x and y, the factors of $1/f_0$ can be pulled out of each derivative. Therefore, (12) simplifies to:

$$\zeta_g = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla_p^2 \Phi \tag{13}$$

In (13), ∇^2 is known as the Laplacian operator. The subscript p denotes that it is computed on an isobaric surface.

Before we proceed, a brief digression about the meaning of (13). It denotes that if you know the geopotential on an isobaric surface, because of geostrophic balance you can obtain the geostrophic relative vorticity and geostrophic winds on that isobaric surface. Conversely, the geostrophic wind (and thus the geostrophic relative vorticity) is known, you can obtain the geopotential field on that isobaric surface. This concept, known as *invertability*, is something that we'll explore further when we discuss potential vorticity.

Furthermore, (13) also gives us insight into the distribution of geostrophic relative vorticity with respect to the geopotential field. The Laplacian operator ∇^2, or second partial derivative, can be viewed as a measurement of the curvature of a field. Nominally, where $\nabla^2\Phi$ is a maximum, Φ itself is a minimum (and vice versa). As a result, maxima in $\nabla^2\Phi$ (and thus also ζ_g) correspond to minima in Φ. Likewise, minima in $\nabla^2\Phi$ (and thus also ζ_g) correspond to maxima in Φ.

Since $\Phi = gz$ and g is a constant, these assessments can be interpreted in terms of the height field rather than just the geopotential…

- Geostrophic relative vorticity is maximized (or is cyclonic) at the base of a trough on an isobaric surface.

- Geostrophic relative vorticity is minimized (or is anticyclonic) at the apex of a ridge on an isobaric surface.

Why are we interested in the geostrophic relative vorticity? The evolution of geostrophic relative vorticity, both in space as well as in time, provides vital information that can be used to diagnose the movement of synoptic-scale meteorological phenomena and the synoptic-scale vertical motion associated with such phenomena. Such information can be used to describe the formation, motion, and evolution of midlatitude cyclones. Much of our discussion of quasi-geostrophic theory follows directly from these tenets, as we will explore in upcoming lectures.

The Quasi-Geostrophic Vorticity Equation

We obtain the quasi-geostrophic vorticity equation by finding *∂/∂x* of the *y* (or **j**) component of the quasi-geostrophic momentum equation and subtracting from it *∂/∂y* of the *x* (or **i**) component of the quasi-geostrophic momentum equation.

Neglecting friction, (5) can be written as:

$$\frac{D_g\vec{\boldsymbol{v}}_g}{Dt} = -f_0\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_{ag} - \beta y\hat{\boldsymbol{k}} \times \vec{\boldsymbol{v}}_g \tag{14}$$

Expanded into its components, (14) becomes:

$$\frac{D_g u_g}{Dt} = f_0 v_{ag} + \beta y v_g \tag{15a}$$

$$\frac{D_g v_g}{Dt} = -f_0 u_{ag} - \beta y u_g \tag{15b}$$

To form the quasi-geostrophic vorticity equation, we thus compute *∂/∂x* of (15b) - *∂/∂y* of (15a). Start by operating on the left-hand side of (15a) and (15b):

$$\frac{\partial}{\partial x}\left(\frac{D_g v_g}{Dt}\right) - \frac{\partial}{\partial y}\left(\frac{D_g u_g}{Dt}\right) = \frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) = \frac{D_g\zeta_g}{Dt} \tag{16a}$$

Now repeat for the right-hand side of (15a) and (15b), setting it equal to the result of (16a):

$$\frac{D_g\zeta_g}{Dt} = \frac{\partial}{\partial x}\left(-f_0 u_{ag} - \beta y u_g\right) - \frac{\partial}{\partial y}\left(f_0 v_{ag} + \beta y v_g\right) \tag{16b}$$

Expanding (16b) and grouping like terms, we obtain:

$$\begin{aligned}\frac{D_g\zeta_g}{Dt} &= -f_0\frac{\partial u_{ag}}{\partial x} - f_0\frac{\partial v_{ag}}{\partial y} - \beta y\frac{\partial u_g}{\partial x} - \beta y\frac{\partial v_g}{\partial y} - \beta v_g \\ &= -f_0\left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y}\right) - \beta y\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) - \beta v_g\end{aligned} \tag{17}$$

Because the divergence of the geostrophic wind, when *f* is held constant ($f = f_0$), is equal to zero, the *–βy* term in (17) will vanish. Likewise, the continuity equation (7) can be used to rewrite the terms involving u_{ag} and v_{ag} in terms of the vertical partial derivative of the vertical motion. Thus, (17) becomes:

$$\frac{D_g\zeta_g}{Dt} = f_0\frac{\partial\omega}{\partial p} - \beta v_g \tag{18}$$

The total derivative on the left-hand side of (19) is expanded as:

$$\frac{D_g \zeta_g}{Dt} = \frac{\partial \zeta_g}{\partial t} + \vec{\boldsymbol{v}}_g \cdot \nabla(\zeta_g) \tag{19}$$

Using (19), (18) may be written as:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{\boldsymbol{v}}_g \cdot \nabla \zeta_g - \beta v_g + f_0 \frac{\partial \omega}{\partial p} \tag{20}$$

Equation (20) is the *quasi-geostrophic vorticity equation*. It is a *prognostic* (or predictive) equation for the geostrophic relative vorticity. It states that the local change of geostrophic relative vorticity is a function of three terms: the geostrophic horizontal advection of geostrophic relative vorticity, the geostrophic meridional advection of planetary vorticity, and the vertical stretching of planetary vorticity.