<u>Model Formulation and Resolved versus Unresolved Scales</u>

Learning Outcomes

Following this lecture, students will be able to:

- Demonstrate that the WRF-ARW model predictive equations are transformed versions of the primitive equations.
- Apply Reynolds averaging to separate resolved from unresolved/sub-grid-scale processes in the primitive equations.

The Equations

Modern NWP models solve the **primitive equations** describing atmospheric motions as well as the conservation of mass, energy, and water vapor. Written in Cartesian coordinates with *z* (height) as the vertical coordinate, the primitive equations are expressed as:

$$\frac{\partial u}{\partial t} = -\mathbf{v}\cdot\nabla u + \frac{uv\tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega\left(w\cos\phi - v\sin\phi\right) + Fr_x \qquad (1)$$

$$\frac{\partial v}{\partial t} = -\mathbf{v}\cdot\nabla v - \frac{u^2\tan\phi}{a} - \frac{vw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + Fr_y \qquad (2)$$

$$\frac{\partial w}{\partial t} = -\mathbf{v}\cdot\nabla w + \frac{u^2 + v^2}{a} - g - \frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi + Fr_z \qquad (3)$$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} + w\left(\Gamma - \Gamma_d\right) + \frac{1}{c_p}\frac{dH}{dt} \qquad (4)$$

$$\frac{\partial \rho}{\partial t} = -\mathbf{v}\cdot\nabla\rho - \rho\left(\nabla\cdot\mathbf{v}\right) \qquad (5)$$

$$\frac{\partial q_v}{\partial t} = -\mathbf{v}\cdot\nabla q_v + Q_v \qquad (6)$$

$$p = \rho RT \qquad (7)$$

In the above, note that **v** is the three-dimensional velocity vector.

Equations (1) through (3) are the momentum equations, themselves being an application of Newton’s second law of motion. The left-hand side terms are local time-rate-of-change terms. The terms on the right-hand side of these equations include advection terms, curvature terms (those involving *a*, the radius of the Earth), pressure gradient terms (those involving *p*), Coriolis terms (those involving Ω), frictional terms (the *Fr* terms), and gravity (those involving *g*).

Equation (4) is a statement of the conservation of energy (e.g., energy is not created or destroyed), manifest as the thermodynamic equation. The left-hand side of this equation is the local time-rate-of-change of temperature, while terms on the right-hand side represent horizontal advection, vertical advection (for $\Gamma = -\frac{\partial T}{\partial z}$), adiabatic cooling/warming (involving Γ_d), and diabatic heating (where *dH/dt* is the diabatic heating rate).

Equation (5) is a statement of the conservation of mass, manifest as the continuity equation; e.g., mass is not created or destroyed. The left-hand side of this equation is the local time-rate-of-change of density, defined as mass per unit volume. On the right-hand side of this equation are advection and divergence terms.

Equation (5) can also be written in **flux form** using the following relationship:

$$-\mathbf{v} \cdot \nabla \rho - \rho(\nabla \cdot \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v})$$

The left-hand side of this relationship includes an advection term and a divergence term, whereas the right-hand side of this relationship includes a flux-divergence term. Such a term represents the flux, or transport, of a quantity into or out of a given volume or area. We will revisit this distinction in the context of the WRF-ARW model governing equations shortly.

Equation (6) is a statement of the conservation of water vapor. The left-hand side of this equation is the local time-rate-of-change of the water vapor mixing ratio. On the right-hand side of this equation is an advection term and a source/sink term Q_v that includes conversions between microphysical species (e.g., rain, cloud water, cloud ice, snow, graupel, and hail).

Full-physics models contain **many** additional equations akin to (6), with one for each microphysical variable that is treated prognostically by the chosen microphysical parameterization used by the model. For example, a model using the WSM6 microphysical parameterization, which predicts mixing ratio for water vapor q_v, rain q_r, cloud water q_c, cloud ice q_i, snow q_s, and graupel/hail q_g, will have six such equations. A model using the WDM6 microphysical parameterization, which also predicts number concentration for rain N_r, cloud water N_c, and cloud condensation nuclei N_{CCN}, will have nine such equations.

The source and sink term in (6), or its counterpart for other microphysical variables, includes both grid-resolved and sub-grid processes. As we will see later this semester, there are *many* processes that must be accounted for by such terms, making equations such as (6) far more complex than they would otherwise seem.

Equation (7) is the ideal-gas law.

In the form presented in (1) – (7), we have seven equations with seven unknowns: *u*, *v*, *w*, *T*, *p*, *ρ*, and q_v. We have **prognostic** (or predictive) equations for six of these unknowns, with pressure obtained from the **diagnostic** ideal-gas law. Thus, with appropriate numerical methods and some means of representing $Fr_{x,y,z}$, *dH/dt*, and Q_v (plus other microphysical source/sink terms), we could solve the primitive equations so as to obtain a forecast valid at some future time.

The Equations Manifest in the WRF-ARW Model

Let us now contrast these equations with their manifestation in the WRF-ARW model (Skamarock et al. 2021, Section 2.2). First, however, a few notes and definitions.

The WRF-ARW model uses a terrain-following vertical coordinate η that is defined primarily as a function of dry hydrostatic pressure (i.e., the pressure of dry air under hydrostatic balance):

$$\eta = \frac{p_{dh} - p_{dht}}{p_0 - p_{dht}} + B(\eta)\left[1 - \frac{p_{dhs} - p_{dht}}{p_0 - p_{dht}}\right] \text{ (where } 0 \le \eta \le 1\text{)}$$

In the above, p_{dh} is the dry hydrostatic pressure, p_{dht} is the dry hydrostatic pressure at the top of the model (generally a user-defined parameter), p_{dhs} is the dry hydrostatic pressure at the surface, p_0 is a reference sea-level pressure, and $B(\eta)$ defines the relative weighting between a purely isobaric vertical coordinate and a terrain-following vertical coordinate. At locations where $p_{dhs} \sim p_0$ (where the surface is near sea-level), the vertical coordinate reduces to the first right-hand side term in the equation above. In all cases, η is 0 at the top of the model (where $p_{dh} = p_{dht}$ and $B(\eta) = 0$) and 1 at the surface (where $p_{dh} = p_{dhs}$ and $B(\eta) = 1$).

The general form of the coordinate transformation between the height and terrain-following vertical coordinates is given by:

$$\nabla_\eta(\;) = \nabla_z(\;) + \frac{\partial(\;)}{\partial z}\nabla_\eta z \qquad \text{(for horizontal coordinate transforms)}$$

$$\frac{\partial(\;)}{\partial \eta} = \frac{\partial z}{\partial \eta}\frac{\partial(\;)}{\partial z} \qquad \text{(for vertical coordinate transforms)}$$

Subscripts on the gradient operator denote the vertical coordinate surface on which it is applied.

For example, converting the *x*-direction pressure gradient term in (1) into the terrain-following vertical coordinate takes the following form:

$$\left.\frac{\partial p}{\partial x}\right|_\eta = \left.\frac{\partial p}{\partial x}\right|_z + \frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\eta$$

Note that the vertical coordinate transformation for advection terms also incorporates the transformation of the vertical velocity from *w* (for constant-height surfaces; change in height with time) to its analogous formulation (change in vertical-coordinate location with time) in the chosen vertical coordinate.

The WRF-ARW model is **coupled** to the dry-air mass field. This means that all variables are multiplied by the hydrostatic dry-air mass per unit area in the column. As pressure is a measure of the air's mass, this hydrostatic dry-air mass can be defined as the change in pressure between two vertical levels; i.e.,

$$\mu_d = \frac{\partial p_{dh}}{\partial \eta}$$

In coupled form, the three-dimensional velocity vector, moist potential temperature, and all mixing ratios take the form:

$$\mathbf{V} = \mu_d \mathbf{v} \qquad \Theta_m = \mu_d \theta_m \qquad Q = \mu_d q$$

The continuity equation for WRF-ARW is written in terms of the hydrostatic dry-air mass, as given by equation (2.12) of Skamarock et al. (2021):

$$\frac{\partial \mu_d}{\partial t} + \nabla \cdot \left(\mu_d \mathbf{v} \right) = 0 \tag{E}$$

Formally, WRF-ARW conserves hydrostatic dry-air mass (rather than total mass), such that this continuity equation is formally the hydrostatic dry-air mass conservation equation. Equation (E) is equivalent to (5), except that it is written in terms of μ_d (hydrostatic dry-air mass) rather than ρ (mass per unit volume), in flux form rather than advective form, and with all terms on the left-hand side of the equation to clearly express the conservative nature of the equation. Here, the local rate of change of hydrostatic dry-air mass is equal to the flux convergence of hydrostatic dry-air mass. Note that (E) can equivalently be written using the coupled velocity vector:

$$\frac{\partial \mu_d}{\partial t} + \nabla \cdot \mathbf{V} = 0 \tag{E.a}$$

We next consider the *u*-momentum equation, given by equation (2.8) of Skamarock et al. (2021):

$$\frac{\partial U}{\partial t} + \nabla \cdot \left(\mathbf{V} u \right) + \mu_d \alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \Phi}{\partial x} = F_u \tag{A}$$

In (A), the first term is the local time-rate-of-change term, the second term is the flux divergence term, the third and fourth terms are pressure gradient terms, and the forcing term on the right-hand side of the equation includes Coriolis, curvature, and frictional terms. $\Phi = gz$ is the geopotential, $\alpha = \rho^{-1}$ is the inverse density, $\alpha_d = \rho_d^{-1}$ is the inverse dry air density, and α is related to α_d by $\alpha = \alpha_d(1+q)^{-1}$, where q refers to the sum of all microphysical species' mixing ratios.

We do not consider the exact forms of the Coriolis and curvature terms at this time because they depend upon the chosen **map projection**, which we cover in more detail in an upcoming lecture.

The most evident way in which (A) differs from (1) is that it is written in terms of the coupled form of *u*, *U*. Thus, to obtain (A), (1) was multiplied by the hydrostatic dry-air mass. Further, we note that (A) contains a flux term whereas (1) contains an advection term. Recall the definition of the flux term, here written in terms of the relevant variables:

$$-\mathbf{V}\cdot\nabla u - u(\nabla\cdot\mathbf{V}) = -\nabla\cdot(\mathbf{V}u)$$

Thus, it is natural to ask: where did the divergence term go? Consider equation (E.a). Solving that equation for the divergence term, we find that it is equal to $-\frac{\partial \mu_d}{\partial t}$. Multiplying this result by *u*, as it appears above, results in $-u\frac{\partial \mu_d}{\partial t}$. Next, by making use of the definition of the coupled velocity vector and the product rule, we can expand $\frac{\partial U}{\partial t}$ as follows:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}(\mu_d u) = u\frac{\partial \mu_d}{\partial t} + \mu_d\frac{\partial u}{\partial t}$$

Note the presence of a $+u\frac{\partial \mu_d}{\partial t}$ term in this expansion. This term exactly balances that from the expansion of the flux divergence term. Thus, if we substitute the above two expansions into (A), we obtain:

$$\mu_d\frac{\partial u}{\partial t} + \mathbf{V}\cdot\nabla u + \mu_d\alpha\frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \Phi}{\partial x} = F_u \quad \text{(A.a)}$$

In (A.a), as in (1), there is a local time-rate-of-change term and an advection term. The only difference in these terms between (A.a) and (1) is that the former is coupled to the hydrostatic dry-air mass. As a result, we state that (A) is equivalent to (1) for these terms.

Note that the *v*-momentum, *w*-momentum, thermodynamic, and mixing ratio conservation equations all involve similar cancellation of a divergence term. We will discuss the forms of these equations shortly.

Continuing with (A), note that (A) contains two pressure gradient terms whereas (1) contains only one such term. This arises because of the vertical coordinate transformation from the *z* coordinate to the η coordinate. To demonstrate this, we start with the form of the pressure gradient term in (1), applicable on constant height surfaces, after substituting $\alpha = \rho^{-1}$:

$$-\alpha\frac{\partial p}{\partial x}_z$$

Apply the vertical coordinate transformation to the partial derivative in this term:

$$\frac{\partial p}{\partial x}_z = \frac{\partial p}{\partial x}_\eta - \frac{\partial p}{\partial z}\frac{\partial z}{\partial x}_\eta \text{, such that } -\alpha\frac{\partial p}{\partial x}_z = -\alpha\frac{\partial p}{\partial x}_\eta + \alpha\frac{\partial p}{\partial z}\frac{\partial z}{\partial x}_\eta$$

Next, substitute for ∂z with the definition of the geopotential, Φ, where $\partial \Phi = g\partial z$:

$$-\alpha \frac{\partial p}{\partial x}_{\eta} + \alpha g \frac{\partial p}{\partial \Phi} \frac{1}{g} \frac{\partial \Phi}{\partial x}_{\eta} \text{, or simply } -\alpha \frac{\partial p}{\partial x}_{\eta} + \alpha \frac{\partial p}{\partial \Phi} \frac{\partial \Phi}{\partial x}_{\eta}$$

We continue to operate on this term using the hydrostatic equation applicable on the model vertical coordinate, which is given by equation (2.15) of Skamarock et al. (2021):

$$\frac{\partial \Phi}{\partial \eta} = -\alpha_d \mu_d \qquad \text{(H)}$$

Equation (H) can be obtained from the hydrostatic equation applicable on constant height surfaces by transforming the vertical coordinate using the appropriate coordinate transform from page three, applying the definitions of the geopotential Φ and hydrostatic dry-air mass μ_d, and simplifying the resulting equation.

If we rearrange (H) to solve for $\partial\Phi$ and plug the result from doing so into the $\frac{\partial p}{\partial \Phi}$ term in the *u*-momentum equation, we obtain:

$$\partial \Phi = -\alpha_d \mu_d \partial \eta \text{, such that we obtain } -\alpha \frac{\partial p}{\partial x}_{\eta} - \frac{\alpha}{\alpha_d} \frac{1}{\mu_d} \frac{\partial p}{\partial \eta} \frac{\partial \Phi}{\partial x}_{\eta}$$

Finally, multiply this equation by μ_d to couple it to the hydrostatic dry-air mass to obtain:

$$-\alpha \mu_d \frac{\partial p}{\partial x}_{\eta} - \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \Phi}{\partial x}_{\eta}$$

Moving these terms from the right-hand side of (1) to the left-hand side gives us the pressure gradient terms as they appear in (A). Thus, we have demonstrated that (1) and (A) are functionally equivalent.

The same principles apply to the *v*- and *w*-momentum equations, given by equations (2.9) and (2.10) of Skamarock et al. (2008):

$$\frac{\partial V}{\partial t} + \nabla \cdot (\mathbf{V} v) + \mu_d \alpha \frac{\partial p}{\partial y} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \Phi}{\partial y} = F_v \qquad \text{(B)}$$

$$\frac{\partial W}{\partial t} + \nabla \cdot (\mathbf{V} w) - g \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} + g \mu_d = F_u \qquad \text{(C)}$$

Note that only one pressure gradient term appears in (C). This is because the vertical coordinate transformation for ∂z only results in one term, whereas those for ∂x and ∂y result in two terms. Specifically,

$$-\alpha\frac{\partial p}{\partial z} = -\alpha\frac{\partial \eta}{\partial z}\frac{\partial p}{\partial \eta} = -g\alpha\frac{\partial \eta}{\partial \Phi}\frac{\partial p}{\partial \eta} = \frac{g\alpha}{\mu_d\alpha_d}\frac{\partial p}{\partial \eta}$$

In the above, we first transformed the vertical coordinate, then applied the definition of the geopotential to the result, then substituted from the hydrostatic equation (H). Multiplying this result by the hydrostatic dry-air mass μ_d and moving it to the left-hand side of the equation results in the pressure gradient term as it appears in (C).

In WRF-ARW, the thermodynamic equation is written in terms of moist potential temperature Θ_m,

$$\Theta_m = \mu_d\theta_m \text{, where } \theta_m = \theta\left(1+\frac{R_v}{R_d}q_v\right) = \theta(1+1.61q_v)$$

where q_v (the water vapor mixing ratio) is dimensionless. For a typical maximum q_v = 40 g kg^{-1} = 0.04, $\theta_m = 1.0644\theta$; in other words, differences between θ and θ_m are generally small. As is θ, θ_m is a conserved quantity for dry adiabatic motions (i.e., conditions in which q_v is constant following the flow). The resulting equation is given by equation (2.11) of Skamarock et al. (2021):

$$\frac{\partial \Theta_m}{\partial t} + \nabla\cdot(\mathbf{V}\theta_m) = F_{\Theta_m} \qquad \text{(D)}$$

Note that Θ_m in the first and last terms of (D) is the coupled moist potential temperature, whereas θ_m in the second term of (D) is the uncoupled moist potential temperature. The first left-hand side term is the local time-rate-of-change for the coupled moist potential temperature whereas the second left-hand side term is the flux form of the horizontal and vertical advection terms for moist potential temperature. The right-hand side of (D) reflects diabatic processes. Thus, absent diabatic processes, (D) simplifies to the conservation statement for θ_m, as coupled to the hydrostatic dry-air mass.

The conservation equation for the various microphysical species is nearly identical to (6) and is given by equation (2.14) of Skamarock et al. (2021):

$$\frac{\partial Q_m}{\partial t} + \nabla\cdot(\mathbf{V}q_m) = F_{Q_m} \qquad \text{(F)}$$

In (F), Q_m is the coupled mixing ratio and q_m is the uncoupled mixing ratio. Here, m is taken to be one of the allowable microphysical species, and there is one equation like (F) for each. The right-hand side of (F) reflects the source/sink term for the given microphysical specie.

WRF-ARW also contains a prognostic equation for the geopotential, given by equation (2.13) of Skamarock et al. (2021). Unlike the other equations, this equation is written in advective form. This equation is obtained by simply taking the total derivative of the geopotential:

$$\Phi = gz \text{, such that } \frac{D\Phi}{Dt} = g\frac{Dz}{Dt}$$

The right-hand side of this equation is equal to *gw*, since *Dz/Dt* is defined as *w*. In terms of the coupled variables, this can be written as:

$$\frac{g}{\mu_d}W$$

The left-hand side of the definition of the geopotential can be expanded into local time rate of change and advection terms by using the definition of the total derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \qquad \text{(in terms of uncoupled fields)}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{\mu_d}\mathbf{V} \cdot \nabla \qquad \text{(in terms of coupled fields)}$$

Making this substitution, in terms of coupled fields, and moving the right-hand side above to the left-hand side, we obtain the prognostic equation for the geopotential:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{\mu_d}\left(\mathbf{V} \cdot \nabla\Phi - gW\right) = 0 \qquad \text{(I)}$$

Note that the geopotential is not a conserved quantity.

The final equation is a diagnostic equation for *p* given by the ideal-gas law. We start with the ideal-gas law for dry air, such that $R = R_d$ and $\rho = \alpha^{-1} = \alpha_d^{-1}$. For now, we do not substitute for temperature with virtual temperature, such that we obtain:

$$p\alpha_d = R_d T$$

Poisson's equation allows us to rewrite *T* in terms of θ:

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}} \text{, such that } T = \theta\left(\frac{p}{p_0}\right)^{\frac{R_d}{c_p}} = \theta p^{\frac{R_d}{c_p}} p_0^{-\frac{R_d}{c_p}}$$

Substituting this expression into the ideal-gas law, we obtain:

$$p\alpha_d = R_d \theta p^{\frac{R_d}{c_p}} p_0^{-\frac{R_d}{c_p}} \text{ or, grouping } p \text{ terms, } pp^{-\frac{R_d}{c_p}} = \frac{R_d}{\alpha_d}\theta p_0^{-\frac{R_d}{c_p}}$$

The left-hand side of this equation can equivalently be written as $p^{1-\frac{R_d}{c_p}}$.

Note that $R_d = c_p - c_v$ (1005.7 J kg^{-1} K^{-1} – 719 J kg^{-1} K^{-1} = 286.7 J kg^{-1} K^{-1}), such that $1-\frac{R_d}{c_p} = 1-\frac{c_p - c_v}{c_p} = 1-1+\frac{c_v}{c_p} = \frac{c_v}{c_p}$ and $-\frac{R_d}{c_p} = -\frac{c_p - c_v}{c_p} = -1+\frac{c_v}{c_p}$. If we let $\gamma = \frac{c_p}{c_v}$, such that $\frac{1}{\gamma} = \frac{c_v}{c_p}$, then $1-\frac{R_d}{c_p} = \frac{1}{\gamma}$ and $-\frac{R_d}{c_p} = -1+\frac{1}{\gamma} = \frac{1-\gamma}{\gamma}$.

If we substitute this into the ideal-gas law, we obtain:

$$p^{\frac{1}{\gamma}} = \frac{R_d}{\alpha_d}\theta p_0^{\frac{1-\gamma}{\gamma}}$$

If we then raise both sides of this expression to the power of γ, we obtain:

$$p = \left(\frac{R_d}{\alpha_d}\theta\right)^{\gamma}\left(p_0^{\frac{1-\gamma}{\gamma}}\right)^{\gamma}$$

Using the properties of exponential functions, we can write:

$$\left(p_0^{\frac{1-\gamma}{\gamma}}\right)^{\gamma} = p_0^{\gamma\frac{1-\gamma}{\gamma}} = p_0^{1-\gamma} = p_0 p_0^{-\gamma}$$

Thus, we obtain:

$$p = \left(\frac{R_d}{\alpha_d}\theta\right)^{\gamma} p_0 p_0^{-\gamma} = p_0\left(\frac{R_d\theta}{\alpha_d p_0}\right)^{\gamma}$$

To account for the effects of moisture upon pressure, we replace θ with θ_m. Earlier, we did not substitute a moisture-based temperature variable in the ideal-gas law when we substituted R_d for R. We now make that substitution, simply replacing θ with θ_m. This results in our final form of the ideal-gas law, given by equation (2.16) of Skamarock et al. (2021):

$$p = p_0\left(\frac{R_d\theta_m}{\alpha_d p_0}\right)^{\gamma} \qquad \text{(G)}$$

We thus have a closed set of nine equations, (A) through (I), with nine unknowns: U, V, W, Θ,

μ_d, Φ, Q_m, α_d, and p. The first seven of these unknowns are prognosed; α_d is diagnosed from Φ and μ_d whereas p is diagnosed from Θ, Q_v, α_d, and μ_d. To first order, these are the equations solved by WRF-ARW to obtain a forecast.

Resolved versus Unresolved Scales

Formally, these equations are valid for all scales of motion. However, because models can only resolve scales of motion above their grid spacings and must parameterize the scales of motion below their grid spacings, it is helpful to reformulate these equations to separate these scales. This is done by performing a *scale separation* on the equations.

For any scalar dependent (or prognostic) variable, the variable can be represented as the sum of mean and perturbation terms, where *mean* refers to the resolved scales (e.g., grid-scale; average over a grid area/volume) and *perturbation* refers to the unresolved scales (e.g., sub-grid-scale):

$$x = \overline{x} + x'$$

We now wish to perform this scale separation on the primitive equations. Let us do so in the context of how they appear in (1) through (7), though we note that the same scale separation can be applied to (A) through (I) as well. The course text illustrates this scale separation for the *u*-momentum equation (1), so we will instead do so for the *v*-momentum equation (2).

We start by expanding the frictional term, which can be written in terms of frictional stresses τ, representing viscous forces in the form of molecular diffusion that arise from molecular motions. Frictional stresses transfer quantities from high toward low values and thus act to homogenize a field. Representing the frictional term Fr_y in (2) in terms of these stresses, we obtain:

$$Fr_y = \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)$$

A given frictional stress $\tau_{\_y}$ represents friction that is exerted on the flow in the *y*-direction – the meridional wind *v* – by the fluid (or molecules) on one side of a constant _-plane as they flow along or across the fluid (or molecules) on the other side of a constant _-plane. This is depicted in Fig. 1 for τ_{xy}.

These frictional stresses can be parameterized as a function of wind shear and a frictional (or dynamic viscosity) coefficient *k*, i.e.,

$$\tau_{xy} = k\frac{\partial v}{\partial x} \qquad \tau_{yy} = k\frac{\partial v}{\partial y} \qquad \tau_{zy} = k\frac{\partial v}{\partial z}$$

Substituting, we obtain:

$$Fr_y = \frac{k}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \frac{k}{\rho}\nabla^2 v$$

In the discussion that follows, however, we will utilize the formulation for Fr_y in terms of τ. We will discuss friction in greater detail when we cover planetary boundary layer, surface layer, and land-surface parameterizations.

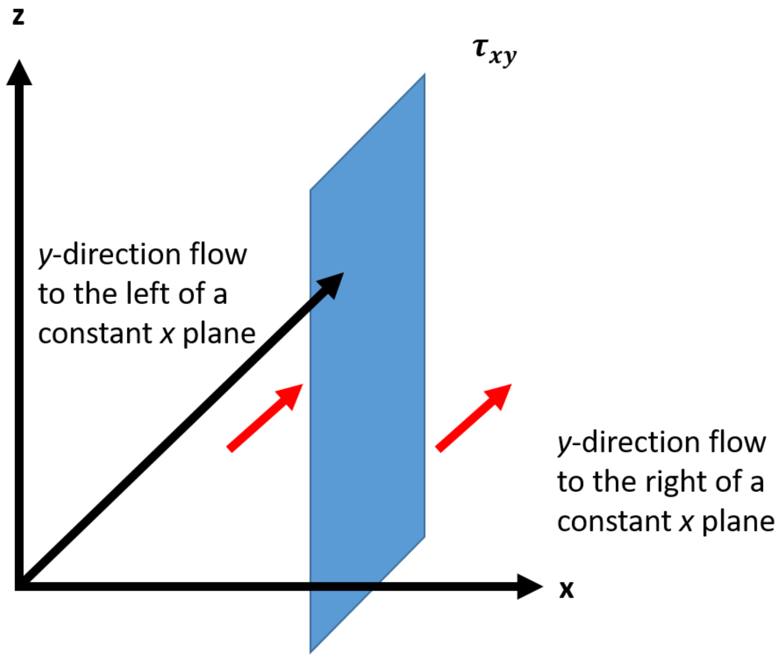

Figure 1. Conceptual illustration of flow on opposite sides of a constant *x* plane (blue; flow vectors in red) comprising the frictional stress τ_{xy}. Frictional stresses relative to constant *y* and *z* planes are similarly construed, except with the constant plane rotated accordingly.

Substituting the τ-based definition for Fr_y into (2), we obtain:

$$\frac{\partial v}{\partial t} = -\mathbf{v}\cdot\nabla v - \frac{u^2 \tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \qquad \text{(a)}$$

The dependent variables in (a) are u, v, w, ρ, p, τ_{xy}, τ_{yy}, and τ_{zy}. If we decompose these into mean (grid-scale-resolved) and perturbation (sub-grid-scale) components, we obtain the following:

$$\frac{\partial(\bar{v}+v')}{\partial t} = -(\bar{\mathbf{v}}+\mathbf{v}')\cdot\nabla(\bar{v}+v') - \frac{(\bar{u}+u')^2\tan\phi}{a} - \frac{(\bar{u}+u')(\bar{w}+w')}{a} - \frac{1}{(\bar{\rho}+\rho')}\frac{\partial(\bar{p}+p')}{\partial y} - 2\Omega(\bar{u}+u')\sin\phi$$
$$+ \frac{1}{(\bar{\rho}+\rho')}\left(\frac{\partial(\overline{\tau_{xy}}+\tau_{xy}')}{\partial x} + \frac{\partial(\overline{\tau_{yy}}+\tau_{yy}')}{\partial y} + \frac{\partial(\overline{\tau_{zy}}+\tau_{zy}')}{\partial z}\right)$$

We assume that $\rho' << \bar{\rho}$ (i.e., density perturbations are small relative to the grid-scale-resolved density). We also note that $f = 2\Omega\sin\phi$. If we expand the above with this in mind, we obtain:

$$
\begin{aligned}
\frac{\partial \bar{v}}{\partial t}+\frac{\partial v'}{\partial t} = & -\bar{u}\frac{\partial \bar{v}}{\partial x}-\bar{u}\frac{\partial v'}{\partial x}-u'\frac{\partial \bar{v}}{\partial x}-u'\frac{\partial v'}{\partial x}-\bar{v}\frac{\partial \bar{v}}{\partial y}-\bar{v}\frac{\partial v'}{\partial y}-v'\frac{\partial \bar{v}}{\partial y}-v'\frac{\partial v'}{\partial y} \\
& -\bar{w}\frac{\partial \bar{v}}{\partial z}-\bar{w}\frac{\partial v'}{\partial z}-w'\frac{\partial \bar{v}}{\partial z}-w'\frac{\partial v'}{\partial z} \\
& -\frac{\tan\phi}{a}\left(\bar{u}\bar{u}+2\bar{u}u'+u'u'\right)-\frac{1}{a}\left(\bar{u}\bar{w}+u'\bar{w}+\bar{u}w'+u'w'\right) \qquad \text{(b)} \\
& -\frac{1}{\bar{\bar{\rho}}}\frac{\partial \bar{p}}{\partial y}-\frac{1}{\bar{\bar{\rho}}}\frac{\partial p'}{\partial y}-f\bar{u}-fu' \\
& +\frac{1}{\bar{\bar{\rho}}}\left(\frac{\partial \overline{\tau_{xy}}}{\partial x}+\frac{\partial \tau_{xy}'}{\partial x}+\frac{\partial \overline{\tau_{yy}}}{\partial y}+\frac{\partial \tau_{yy}'}{\partial y}+\frac{\partial \overline{\tau_{zy}}}{\partial z}+\frac{\partial \tau_{zy}'}{\partial z}\right)
\end{aligned}
$$

To make (b) applicable only on the grid-scales of motion and larger, we take what is known as the **Reynolds average** of the equation. This involves taking the *mean* of the entire equation. Note that this should *not* be confused with linearizing the equation, as is frequently done when studying wave solutions in dynamic meteorology. While in many ways similar, we want to retain the non-linearity inherent to the primitive equations.

There are four Reynolds' postulates that help us when taking Reynolds averages:

- $\overline{a'} = 0$ (the grid-scale mean of all sub-grid perturbations is zero)
- $\bar{\bar{a}} = \bar{a}$ (the mean of a mean is equivalent to the mean)
- $\overline{\bar{a}b} = \bar{\bar{a}}\bar{b} = \bar{a}\bar{b}$ (similar to the previous postulate, except including a second variable)
- $\overline{\bar{a}b'} = \bar{a}\overline{b'} = \bar{a}0 = 0$ (because of the first postulate)

Of these, the first and last postulates are the most important, as they result in many terms in (b) becoming zero when taking the Reynolds average. However, note that $\overline{a'b'} \neq 0$. Terms such as this are known as **covariance** terms and are only zero if the perturbation fields are not correlated (nominally, not physically correlated) with each other. In practice, we do not have advanced knowledge of whether these correlations are or should be zero, so we do not neglect them.

Taking the Reynolds average of (b) and using the Reynolds postulates to simplify the result, we obtain:

$$\frac{\partial \bar{v}}{\partial t} = -\bar{u}\frac{\partial \bar{v}}{\partial x} - \overline{u'\frac{\partial v'}{\partial x}} - \bar{v}\frac{\partial \bar{v}}{\partial y} - \overline{v'\frac{\partial v'}{\partial y}} - \bar{w}\frac{\partial \bar{v}}{\partial z} - \overline{w'\frac{\partial v'}{\partial z}}$$
$$-\frac{\tan\phi}{a}\left(\bar{u}\bar{u} + \overline{u'u'}\right) - \frac{1}{a}\left(\bar{u}\bar{w} + \overline{u'w'}\right) - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y} - f\bar{u} \quad \text{(c)}$$
$$+\frac{1}{\bar{\rho}}\left(\frac{\partial \overline{\tau_{xy}}}{\partial x} + \frac{\partial \overline{\tau_{yy}}}{\partial y} + \frac{\partial \overline{\tau_{zy}}}{\partial z}\right)$$

We can rewrite the second, fourth, and sixth terms on the right-hand side of (c) in flux form. This is a bit simpler than it was before, however. On the sub-grid (i.e., turbulence) scale, the following continuity equation applies on constant height surfaces:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = \nabla \cdot \mathbf{v}' = 0$$

Multiplying this equation by *–v'* and taking the Reynolds average, we obtain:

$$-\overline{v'\frac{\partial u'}{\partial x}} - \overline{v'\frac{\partial v'}{\partial y}} - \overline{v'\frac{\partial w'}{\partial z}} = 0 \text{, which can be written as } \overline{-v'\left(\nabla \cdot \mathbf{v}'\right)} = 0$$

This is a divergence term and, as stated above, is equal to zero on these scales. Conversely, the second, fourth, and sixth terms on the right-hand side of (c) comprise an advection term of the form $-\overline{\mathbf{v}' \cdot \nabla v'}$. Recalling that a flux term is equal to the advection term plus the divergence term (which, here, is equal to zero, and thus can be added without changing the result), the advection terms can be equivalently written as flux terms:

$$-\frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y} - \frac{\partial \overline{w'v'}}{\partial z}$$

Further, we can define *turbulent stresses* (i.e., the grid-scale-resolved effects of parameterized sub-grid processes; note the different definition from the frictional stresses defined earlier) as:

$$T_{xy} = -\bar{\rho}\overline{u'v'} \qquad T_{yy} = -\bar{\rho}\overline{v'v'} \qquad T_{zy} = -\bar{\rho}\overline{w'v'}$$

Substituting these definitions for the flux terms and combining the result with the frictional stresses, we obtain:

$$\frac{\partial \bar{v}}{\partial t} = -\bar{u}\frac{\partial \bar{v}}{\partial x} - \bar{v}\frac{\partial \bar{v}}{\partial y} - \bar{w}\frac{\partial \bar{v}}{\partial z} - \frac{\tan\phi}{a}\left(\bar{u}\bar{u} + \overline{u'u'}\right) - \frac{1}{a}\left(\bar{u}\bar{w} + \overline{u'w'}\right) - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y} - f\bar{u}$$
$$+\frac{1}{\bar{\rho}}\left(\frac{\partial\left(T_{xy} + \overline{\tau_{xy}}\right)}{\partial x} + \frac{\partial\left(T_{yy} + \overline{\tau_{yy}}\right)}{\partial y} + \frac{\partial\left(T_{zy} + \overline{\tau_{zy}}\right)}{\partial z}\right) \quad \text{(d)}$$

From left to right, the terms of (d) are the local time-rate-of-change (on the grid scale only, as this is what the model predicts), advection terms, curvature terms, a pressure gradient term, a Coriolis term, and turbulent and frictional stresses. All terms of the form $\overline{a'b'}$ are parameterized. The $\overline{\tau_{\_y}}$ terms, which explicitly refer to sub-grid-scale friction that is a function of molecular motion, are also parameterized. Other terms not involving perturbations are resolved on the model grid.

Generally, (d) is written in a form that drops the resolved-scale notation, e.g.,

$$\frac{\partial v}{\partial t} = -\mathbf{v} \cdot \nabla v - \frac{u^2 \tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_v \qquad \text{(e)}$$

where F_v represents all sub-grid-scale processes, as in (B). Otherwise, (e) closely resembles (2). The remaining equations (1) and (3) through (7) can be transformed similarly.